Operations Research

Course Description:
Operations Research is a very important area of study, which tracks its roots to business applications. It combines the three broad disciplines of Mathematics, Computer Science, and Business Applications. This course will formally develop the ideas of developing, analyzing, and validating mathematical models for decision problems, and their systematic solution. The course will involve programming and mathematical analysis.

Course Objectives:
Upon completion of this course, the students will be able to:

- Solve business problems and apply its applications by using computer programming and mathematical analysis.
- Develop the ideas of developing, analyzing, and validating mathematical models for decision problems, and their systematic solution.
- Understand the main concepts of OR.

Unit-I

Introduction
The term operations research was first coined in 1940 by McClosky and Trefthen in a small town Bowdsey, of the United Kingdom. This new science came into existence in a military context. During World War II, military management called on scientists from various disciplines and organized them into teams to assist in solving strategic and tactical problems, relating to air and land defense of the country. Their mission was to formulate specific proposals and plans for aiding the Military commands to arrive at decisions on optimal utilization of scarce military resources and efforts and also to implement the decisions effectively. This new approach to the systematic and scientific study of the operations of the system was called Operations Research or operational research. Hence OR can be associated with "an art of winning the war without actually fighting it."

Scope of Operations Research
There is a great scope for economists, statisticians, administrators and the technicians working as a team to solve problems of defence by using the OR approach. Besides this, OR is useful in the various other important fields like:
1. Agriculture.
2. Finance.
3. Industry.
5. Personnel Management.
6. Production Management.
7. Research and Development.
Phases of Operational Research
The procedure to be followed in the study of OR, generally involves the following major phases.
1. Formulating the problem.
2. Constructing a mathematical model.
3. Deriving the solution from the model.
4. Testing the model and its solution (updating the model).
5. Controlling the solution.
6. Implementation.

Models in Operations Research
A model in OR is a simplified representation of an operation, or is a process in which only the basic aspects or the most important features of a typical problem under investigation are considered. The objective of a model is to identify significant factors and interrelationships. The reliability of the solution obtained from a model, depends on the validity of the model representing the real system. A good model must possess the following characteristics:
(i) It should be capable of taking into account, new formulation without having any changes in its frame.
(ii) Assumptions made in the model should be as small as possible.
(iii) Variables used in the model must be less in number ensuring that it is simple and coherent.
(iv) It should be open to parametric type of treatment.
(v) It should not take much time in its construction for any problem.

Advantages of a Model
There are certain significant advantages gained when using a model these are:
(i) Problems under consideration become controllable through a model.
(ii) It provides a logical and systematic approach to the problem.
(iii) It provides the limitations and scope of an activity.
(iv) It helps in finding useful tools that eliminate duplication of methods applied to solve problems.
(v) It helps in finding solutions for research and improvements in a system.
(vi) It provides an economic description and explanation of either the operation, or the systems they represent.

Models by Structure
Mathematical models are most abstract in nature. They employ a set of mathematical symbols to represent the components of the real system. These variables are related together by means of mathematical equations to describe the behavior of the system. The solution of the problem is then obtained by applying well-developed mathematical techniques to the model.

BASIC OR CONCEPTS
"OR is the representation of real-world systems by mathematical models together with the use of quantitative methods (algorithms) for solving such models, with a view to optimizing."
We can also define a mathematical model as consisting of:
- Decision variables, which are the unknowns to be determined by the solution to the model.
- Constraints to represent the physical limitations of the system
- An objective function
- An optimal solution to the model is the identification of a set of variable values which are feasible (satisfy all the constraints) and which lead to the optimal value of the objective function.

An optimization model seeks to find values of the decision variables that optimize (maximize or minimize) an objective function among the set of all values for the decision variables that satisfy the given constraints.

**TERMINOLOGY**

**Solution:** The set of values of decision variables \( j = 1, 2, \ldots, n \) which satisfy the constraints is said to constitute solution to meet problem.

**Feasible Solution:** The set of values of decision variables \( X_j (j = 1, 2, \ldots, n) \) which satisfy all the constraints and non-negativity conditions of an linear programming problem simultaneously is said to constitute the feasible solution to that problem.

**Infeasible Solution:** The set of values of decision variables \( X_j (j = 1, 2, \ldots, n) \) which do not satisfy all the constraints and non-negativity conditions of the problem is said to constitute the infeasible solution to that linear programming problem.

**Basic Solution:** For a set of \( m \) simultaneous equations in \( n \) variables \( (n > m) \), a solution obtained by setting \( (n - m) \) variables equal to zero and solving for remaining \( m \) variables is called a “basic feasible solution”.

The variables which are set to zero are known as non-basic variables and the remaining \( m \) variables which appear in this solution are known as basic variables:

**Basic Feasible Solution:** A feasible solution to LP problem which is also the basic solution is called the “basic feasible solution”. Basic feasible solutions are of two types;

(a) Degenerate: A basic feasible solution is called degenerate if value of at least one basic variable is zero.

(b) Non-degenerate: A basic feasible solution is called ‘non-degenerate’ if all values of \( m \) basic variables are non-zero and positive.

**Optimum Basic Feasible Solution:** A basic feasible solution which optimizes (maximizes or minimizes) the objective function value of the given LP problem is called an ‘optimum basic feasible solution’.

**Unbounded Solution:** A basic feasible solution which optimizes the objective function of the LP problem indefinitely is called ‘unbounded solution’.

**Introduction of Linear Programming**

Linear programming deals with the optimization (maximization or minimization) of a function of variables known as objective functions. It is subject to a set of linear equalities and/or inequalities known as *constraints*. Linear programming is a mathematical technique, which involves the allocation of limited resources in an optimal manner, on the basis of a given criterion of optimality.
In this section properties of Linear Programming Problems (LPP) are discussed. The graphical method of solving a LPP is applicable where two variables are involved. The most widely used method for solving LP problems consisting of any number of variables is called Simplex method.

Formulation of LP Problems

The procedure for mathematical formulation of a LPP consists of the following steps:

*Step 1* To write down the decision variables of the problem.

*Step 2* To formulate the objective function to be optimized (Maximized or Minimized) as a linear function of the decision variables.

*Step 3* To formulate the other conditions of the problem such as resource limitation, market constraints, interrelations between variables etc., as linear in equations or equations in terms of the decision variables.

*Step 4* To add the non-negativity constraint from the considerations so that the negative values of the decision variables do not have any valid physical interpretation.

The objective function, the set of constraint and the non-negative constraint together form a Linear programming problem.

General Formulation of LPP

The general formulation of the LPP can be stated as follows:

In order to find the values of \( n \) decision variables \( x_1, x_2, x_3, \ldots, x_n \) to maximize or minimize the objective function.

\[
Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \quad (1)
\]

and also satisfy \( m \) constraints

\[
a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \leq b_1 \\
a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n \geq b_2 \\
\vdots \\
a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \leq b_m
\]

where constraints may be in the form of inequality < or > or even in the form an equation (=) and finally satisfy the nonnegative restrictions

\[
x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0, \ldots, x_n \geq 0.
\]

**Ex:** A manufacturer produces two types of models M1 and M2. Each model of the type M1 requires 4 hrs of grinding and 2 hours of polishing; whereas each model of the type M2 requires 2 hours of grinding and 5 hours of polishing. The manufacturers have 2 grinders and 3 polishers. Each grinder works 40 hours a week and each polisher works for 60 hours a week. Profit on M1 model is Rs.3.00 and on model M2, is Rs.4.00. Whatever is produced in a week is sold in the market. How should the manufacturer allocate his production capacity to the two types of models, so that he may make the maximum profit in a week?

**Solution:**

**Decision Variables:** Let \( x_1 \) and \( x_2 \) be the number of units of M1 and M2 model.
Objective function: Since the profit on both the models are given, we have to maximize the profit
Max (Z) = 3x_1 + 4x_2

Constraints: There are two constraints one for grinding and the other for polishing.
No.of hrs. Available on each grinder for one week is 40 hrs.
There are 2 grinders.
Hence the manufacturer does not have more than 2 * 40 = 80 hrs of grinding. M1 requires 4 hrs of grinding and M2, requires 2 hours of grinding.
The grinding constraint is given by
4x_1 + 2x_2 \leq 80
Since there are 3 polishers, the available time for polishing in a week is given by 3 x 60= 180.
Hence, we have 2x_1 + 5x_2 \leq 180
Finally we have
Max (Z) = 3x_1 + 4x_2
Subject to 4x_1 + 2x_2 \leq 80
\hspace{0.5cm} 2x_1 + 5x_2 \leq 180

**GRAPHICAL METHOD:**

If there are two variables in an LP problem, it can be solved by graphical method. Let the two variables be x_1 and x_2. The variable x_1 is represented on x-axis and x_2 on y-axis. Due to non-negativity condition, the variables x_1 and x_2 can assume positive values and hence the if at all solution exists for the problem, it lies in the first quadrant. The steps used in graphical method are summarized as follows:

**Step 1:** Replace the inequality sign in each constraint by an equal to sign.

**Step 2:** Represent the first constraint equation by a straight line on the graph. Any point on this line satisfies the first constraint equation. If the constraint inequality is ‘\_’ type, then the area (region) below this line satisfies this constraint. If the constraint inequality is of ‘\_’ type, then the area (region) above the line satisfies this constraint.

**Step 3:** Repeat step 2 for all given constraints.

**Step 4:** Shade the common portion of the graph that satisfies all the constraints simultaneously drawn so far. This shaded area is called the ‘feasible region’ (or solution space) of the given LP problem. Any point inside this region is called feasible solution and provides values of x_1 and x_2 that satisfy all constraints.

**Step 5:** Find the optimal solution for the given LP problem. The optimal solution may be determined using the following methods.

**Extreme Point Method:** In this method, the coordinates of each extreme point are substituted in the objective function equation, whichever solution has optimum value of Z (maximum or minimum) is the optimal solution.

**Iso-Profit (Cost) Function Line Method:** In this method the objective function is represented by a straight line for arbitrary value of Z. Generally the objective function line is represented by a dotted line to distinguish it from constraint lines. If the objective is to maximize the value of Z, then move the objective function line parallel till it touches the farthest extreme point. This
farthest extreme point gives the optimum solution. If objective is to minimize the value of \( Z \), then move the objective function line parallel until it touches the nearest extreme point. This nearest extreme point gives the optimal solution.

**Ex:** Solve the following LPP using graphical method.

Maximize \( Z = 3 \, x_1 + 4 \, x_2 \)

Subject to \( x_1 + x_2 \leq 450 \)
\( 2 \, x_1 + x_2 \leq 600 \)
\( x_1, x_2 \geq 0 \)

**Solution:**

Represent \( x_1 \), on \( x \)-axis and \( x_2 \) on \( y \)-axis. Represent the constraints on \( x \) and \( y \) axis to an appropriate scale as follows:

Replace the inequality sign of the first constraint by an equality sign, the first constraint becomes
\( x_1 + x_2 = 450 \)

This can be represented by a straight line \( x \)

If \( x_1 = 0 \), then \( x_2 = 450 \)

If \( x_2 = 0 \), then \( x_1 = 450 \)

The straight line of the first constraint equation passes through coordinates (0, 450) and (450, 0) as shown in Fig.

Any point lying on this line satisfies the constraint equation. Since the constraint is inequality of ‘_’ type, to satisfy this constraint inequality the solution must lie towards left of the line. Hence mark arrows at the ends of this line to indicate to which side the solution lies.

Replace the inequality sign of the second constraint by an equality sign, then the constraint equation is
\( 2 \, x_1 + x_2 = 600 \)

This constraint equation can be represented by straight line

If \( x_1 = 0 \) then \( x_2 = 600 \)

If \( x_2 = 0 \) then \( x_1 = 300 \)
The line passes through co-ordinates (0, 600) and (300,0) as shown in Fig. Any point lying on this straight line satisfies the constraint equation. Since the constraint is inequality of ‘_’ type, the solution should lie towards left of the line to satisfy this constraint inequality. The shaded area shown in figure satisfies both the constraints as well as non-negative condition. This shaded area is called the ‘solution space or the region of feasible solutions’.

To Find Optimal Solution

Extreme Point’s Method: Name the corners or extreme points of the solution space. The solution space is OABCO. Find the coordinates of each extreme point, substitute them in objective function equation and find the value of Z as below:

<table>
<thead>
<tr>
<th>Extreme point</th>
<th>Coordinates</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>A</td>
<td>300</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>150</td>
<td>300</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>450</td>
</tr>
</tbody>
</table>

Extreme point ‘C’ gives the maximum value of Z. Hence, the solution is x1 = 0, x2 = 450 and maximum Z = 1800.

Unbounded Solution:

Some of the LP problem may not have a finite solution. The values of one or more decision variables and the value of the objective function are permitted to increase infinitely without violating the feasibility condition, then the solution is said to be unbounded. It is to be noted that the solution may be unbounded for maximization type of objective function. This is because in minimization type of objective function, the lower boundary is formed by non-negative condition for decision variables.

Ex: Solve the following LPP using graphical method.
Max Z = 3 x1 + 2 x2
Subject to
\[ x_1 - x_2 \geq 1 \]
\[ x_1 + x_2 \geq 3 \]
\[ 4.5x_1 + 3 x_2 \geq 3 \]
\[ x_1, x_2 \geq 0 \]

Solution: \( x_1 - x_2 = 1 \)
If \( x_1 = 0, x_2 = -1 \)
If \( x_2 = 0, x_1 = 1 \)
\( x_1 - x_2 = 1 \) passes through (0, -1) (1, 0)
\( x_1 + x_2 = 3 \)
If \( x_1 = 0, x_2 = 3 \)
If \( x_2 = 0, x_1 = 3 \)
\( x_1 + x_2 = 3 \) passes through (0, 3) (3, 0)
4.5x_1 + 3x_2 \geq 3 \\
\text{If } x_1 = 0, \ x_2 = 3 \\
\text{If } x_2 = 0, \ x_2 = 2 \\
4.5x_1 + 3x_2 \geq 3 \text{ passes through } (0, 3) \ (2, 0) \\
The solution space is in Fig. \\
Assume \ Z = 6 \text{ then } 6 = 3 \ x_1 + 2 \ x_2 \text{ passes through } (0, 3) \ (2, 0). \\
Move the objective function line till it touches the farthest extreme point of solution space. Since there is no closing boundary for the solution space, the dotted line can be moved to infinity. That is \ Z \text{ will be maximum at infinite values of } x_1 \text{ and } x_2. \text{ Hence the solution is unbounded.} \\
\textbf{Note:} \text{ In an unbounded solution, it is not necessary that all the variables can be made arbitrarily large as } Z \text{ approaches infinity. In the above problem if the second constraint is replaced by } x_1 \leq 2, \text{ then, only } x_1 \text{ can approach infinity and } x_1 \text{ cannot be more than two.} \\

\textbf{Infeasible Solution:} \\

Infeasibility is a condition when constraints are inconsistent (mutually exclusive) \textit{i.e.,} no value of the variable satisfy all of the constraints simultaneously. There is not unique (single) feasible region. It should be noted that infeasibility depends solely on constraints and has nothing to do with the objective function. \\
\textbf{Ex:} \\
\begin{align*}
\text{Maximize } Z &= 3 \ x_1 - 2 \ x_2 \\
\text{Subject to } & x_1 + x_2 \geq 1 \\
& 2 \ x_1 + 2 \ x_2 \leq 4 \\
& x_1, x_2 \geq 0 \\
\end{align*} \\
\textbf{Solution:} \ x_1 + x_2 = 1 \text{ passes through } (0, 1) \ (1, 0) \\
2 \ x_1 + 2 \ x_2 = 4 \text{ passes through } (0, 2) \ (2, 0) \\
To satisfy the first constraint the solution must lie to the left of line } AB. \text{ To satisfy the second constraint the solution must lie to the right of line } CD. \text{ There is no point } (x_1, x_2) \text{ which satisfies both the constraints simultaneously. Hence, the problem has no solution because the constraints are inconsistent.
Note: The geometric method of solving linear programming problems presented before. The graphical method is useful only for problems involving two decision variables and relatively few problem constraints.

What happens when we need more decision variables and more problem constraints?
We use an algebraic method called the simplex method, which was developed by George B. DANTZIG (1914-2005) in 1947 while on assignment with the U.S. Department of the air force.

Simplex Method

Most real-world linear programming problems have more than two variables and thus are too complex for graphical solution. A procedure called the simplex method may be used to find the optimal solution to multivariable problems. The simplex method is actually an algorithm (or a set of instructions) with which we examine corner points in a methodical fashion until we arrive at the best solution—highest profit or lowest cost. Computer programs and spreadsheets are available to handle the simplex calculations for you. But you need to know what is involved behind the scenes in order to best understand their valuable outputs.

Summary of the Simplex Method
- Add slack variables to change the constraints into equations and write all variables to the left of the equal sign and constants to the right.
- Write the objective function with all nonzero terms to the left of the equal sign and zero to the right. The variable to be maximized must be positive.
- Set up the initial simplex tableau by creating an augmented matrix from the equations, placing the equation for the objective function last.
- Determine a pivot element and use matrix row operations to convert the column containing the pivot element into a unit column.
- If negative elements still exist in the bottom row, repeat Step 4. If all elements in the bottom row are positive, the process has been completed.
- When the final matrix has been obtained, determine the final basic solution. This will give the maximum value for the objective function and the values of the variables where this maximum occurs.

Example

Maximize: \( P = 3x + 4y \) subject to:
\[ x + y \leq 4 \\
2x + y \leq 5 \\
x \geq 0, \ y \geq 0 \]

Our first step is to classify the problem. Clearly, we are going to maximize our objective function, all are variables are nonnegative, and our constraints are written with our variable combinations less than or equal to a constant. So this is a standard maximization problem and we know how to use the simplex method to solve it.

We need to write our initial simplex tableau. Since we have two constraints, we need to introduce the two slack variables \( u \) and \( v \). This gives us the equalities
\[
\begin{align*}
x + y + u &= 4 \\
2x + y + v &= 5
\end{align*}
\]

We rewrite our objective function as \(-3x - 4y + P = 0\) and from here obtain the system of equations
\[
\begin{align*}
x + y + u &= 4 \\
2x + y &= 5 \\
-3x - 4y + P &= 0
\end{align*}
\]

This gives us our initial simplex tableau:

\[
\begin{array}{ccccc|c}
 x & y & u & v & P \\
\hline
1 & 1 & 1 & 0 & 0 & 4 \\
2 & 1 & 0 & 1 & 0 & 5 \\
-3 & -4 & 0 & 0 & 1 & 0 \\
\end{array}
\]

To find the column, locate the most negative entry to the left of the vertical line (here this is \(-4\)).
To find the pivot row, divide each entry in the constant column by the entry in the corresponding in the pivot column. In this case, we’ll get 4/1 as the ratio for the first row and 5/1 for the ratio in the second row. The pivot row is the row corresponding to the smallest ratio, in this case 4. So our pivot element is the in the second column, first row, hence is 1.

Now we perform the following row operations to get convert the pivot column to a unit column

\[
\frac{R_2}{R_2} \quad \frac{R_2 - R_1}{R_2} \\
\frac{R_3}{R_3} \quad \frac{R_3 + 4R_1}{R_3}
\]

Our simplex tableau has transformed into

\[
\begin{array}{ccccc|c}
 x & y & u & v & P \\
\hline
1 & 1 & 1 & 0 & 0 & 4 \\
1 & 0 & -1 & 1 & 0 & 1 \\
1 & 0 & 4 & 0 & 1 & 16 \\
\end{array}
\]
Notice that all of the entries to the left of the vertical line in the last row are nonnegative.
We must stop here!
To read off the solution from the table, first find the unit columns in the table. The variables
above the unit columns are assigned the value in the constant column in the row where the 1 is in
the unit column. Every variable above a non-unit column is set to 0. So here y = 4, v = 1, P = 16,
x = 0, and u = 0.
Thus, our maximum occurs when x = 0, y = 4 and the maximum value is 16.

**Artificial Variables Techniques**

**INTRODUCTION**

LPP in which constraints may also have $\geq$ and = signs after ensuring that all $b_i \geq 0$ are considered
in this section. In such cases basis of matrix cannot be obtained as an identity matrix in the
starting simplex table, therefore we introduce a new type of variable called the \textit{artificial variable}.
These variables are fictitious and cannot have any physical meaning. The artificial variable
 technique is a device to get the starting basic feasible solution, so that simplex procedure may be
adopted as usual until the optimal solution is obtained. To solve such LPP there are two methods.
(i) The Big $M$ Method or Method of Penalties.
(ii) The Two-phase Simplex Method.

**THE BIG M METHOD**
The following steps are involved in solving an LPP using the Big M method.

**Step 1** Express the problem in the standard form.
**Step 2** Add non-negative artificial variables to the left side of each of the equations
 corresponding to constraints of the type $\geq$ or = However, addition of these artificial variable
 causes violation of the corresponding constraints. Therefore, we would like to get rid of these
variables and would not allow them to appear in the final solution. This is achieved by assigning
a very large penalty (-M for maximization and + M for minimization) in the objective function.
**Step 3** Solve the modified LPP by simplex method, until anyone of the three cases may arise.

**BIG M METHOD CRITERION: OPTIMALITY**
The optimal solution to the augmented problem to the original problem if there are no artificial
variables with non zero value in the optimal solution.

**BIG M METHOD CRITERION: NO FEASIBLE SOLUTION**
If any artificial variable is in the basis with nonzero value at the optimal solution of the
augmented problem, then the original problem has no feasible solution. The solution satisfies the
constraints but does not optimize the objective function, since it contains a very large penalty M
and is called \textit{pseudo optimal solution}.

**BIG M METHOD CRITERION: DEGENERATE SOLUTION**
If at least one artificial variable in the basis at zero level and the optimality condition is satisfied
then the current solution is an optimal basic feasible solution (though degenerated solution).
**Note:** While applying simplex method, whenever an artificial variable happens to leave the basis,
we drop that artificial variable and omit all the entries corresponding to its column from the
simplex table.
Ex:
Use Big M method to Maximize \( z = 3x_1 + 2x_2 \) Subject to the constraints
\( 2x_1 + x_2 \leq 2 \)
\( 3x_1 + 4x_2 \geq 2 \)
\( x_1, x_2 \geq 0 \)

**SOLUTION**

**Step 1. Express the problem in standard form**
Slack variables \( s_1 \) and \( s_2 \) are add and subtracted from the left-hand sides of the constraint 1 and 2 respectively to convert them to equations. These variable \( s_2 \) is called negative slack variable or surplus variable. Variable \( s_1 \) represents excess of availability of 2 units on constraint 1, \( s_2 \) represents excess of requirement of 12 on constraint 2. Since they represent 'free', the cost/profit coefficients associated with them in the objective function are zeros. The problem, therefore, can be written as follows
Maximize \( Z = 3x_1 + 2x_2 + 0s_1 + 0s_2 \)

\[
2x_1 + x_2 + s_1 = 2 \\
3x_1 + 4x_2 - s_2 = 2 \\
x_1, x_2, s_1, s_2 \geq 0
\]

**Step 2. Find initial basic feasible solution**
Putting \( x_1 = x_2 = 0 \), we get \( s_1=2, s_2 =-12 \) as the first basic solution but it is not feasible as \( s_2 \) have negative values that do not satisfy the non-negativity restrictions. Therefore, we introduce artificial variables \( A_1 \) in the second constraint, which take the form
\[
2x_1 + x_2 + s_1 = 2 \\
3x_1 + 4x_2 - s_2 + A_1 = 2 \\
x_1, x_2, s_1, s_2, A_1 \geq 0
\]

Now artificial variables with values greater than zero violate the equality in constraints established in step1. Therefore, \( A_1 \) should not appear in the final solution. To achieve this, they are assigned a large unit penalty (a large negative value, - M) in the objective function, which can be written as Maximize \( z = 3x_1 + 2x_2 + 0s_1 + 0s_2 - MA_1 \)

Subject to \( 2x_1 + x_2 + s_1 = 2 \)
\[
3x_1 + 4x_2 - s_2 + A_1 = 2 \\
x_1, x_2, s_1, s_2, A_1 \geq 0
\]

Eliminating 1 \( A \) from the second equation, modified objective function can be written as
Maximize \( z = (3 +3M)x_1 + (2 + 4M)x_2 + 0s_1 - M s_2 -12M \)

Let \( M =50 \) as default value, then, we have
Maximize \( z = 153x_1 + 202x_2 + 0s -5s_2 - 600 \)
Problem, now, has five variables and two constraints. Three of the variables have to be zeroised to get initial basic feasible solution to the 'artificial system'.
Putting \( x_1 = x_2 = s_2 = 0 \), we get
The starting feasible solution is \( s_1 = 2, A_1 = 12 \) and \( z = -600 \)
Note that we are starting with a very heavy negative profit which we shall maximize during the solution.

First iteration simplex tableau

To improve this solution, we determine that \( x_2 \) is the entering variable, because \(-202\) is the smallest entry in the bottom row.

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( s_2 )</th>
<th>( s_1 )</th>
<th>( A_1 )</th>
<th>( b )</th>
<th>( \text{B.V.} )</th>
<th>( \text{Ratio} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>( s_1 ) Leaving variable 2/1=2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>12</td>
<td>( A_2 ) 12/4=3</td>
<td></td>
</tr>
<tr>
<td>-153</td>
<td>-202</td>
<td>50</td>
<td>0</td>
<td>0</td>
<td>-600</td>
<td>( z )</td>
<td></td>
</tr>
</tbody>
</table>

Second iteration simplex tableau

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( s_2 )</th>
<th>( s_1 )</th>
<th>( A_1 )</th>
<th>( b )</th>
<th>( \text{B.V.} )</th>
<th>( \text{Ratio} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>( x_2 ) 2/1=2</td>
<td></td>
</tr>
<tr>
<td>-5</td>
<td>0</td>
<td>-1</td>
<td>-4</td>
<td>1</td>
<td>12</td>
<td>( A_2 ) 12/4=3</td>
<td></td>
</tr>
<tr>
<td>251</td>
<td>0</td>
<td>50</td>
<td>202</td>
<td>0</td>
<td>-196</td>
<td>( z )</td>
<td></td>
</tr>
</tbody>
</table>

Since all \( Z \) coefficients and an artificial variable appears in the objective row at positive level, the solution of given LPP does not possess any feasible solution.

THE TWO PHASE METHOD
In the preceding section we observed that it was frequently necessary to add artificial variables to the constraints to obtain an initial basic feasible solution to an L.P. problem. If the problem is to be solved, the artificial variables must be driven to zero. The two-phase method is another method to handle these artificial variables. Here the L.P. problem is solved in two phases.

PHASE I
In this phase we find an initial basic feasible solution to the original problem. For this all artificial variables are to be driven to zero. To do this an artificial (Auxiliary) objective function \( (r) \) is created which is the sum of all the artificial variables. This new objective function is then
minimized, subject to the constraints of the given (original) problem, using the simplex method. At the end of phase I, two cases arise:

**TWO PHASE METHOD CRITERION: NO FEASIBLE SOLUTION**

If the minimum value of $r > 0$, and at least one artificial variable appears in the basis at a positive level, then the given problem has no feasible solution and the procedure terminates.

**TWO PHASE METHOD CRITERION: OPTIMALITY**

If the minimum value of $r = 0$, and no artificial variable appears in the basis, then a basic feasible solution to the given problem is obtained. The artificial column (s) are deleted for phase II computations. If the minimum value of $r = 0$ and one or more artificial variables appear in the basis at zero level, then a feasible solution to the original problem is obtained. However, we must take care of this artificial variable and see that it never becomes positive during phase II computations. Zero cost coefficient is assigned to this artificial variable and it is retained in the initial table of phase II. If this variable remains in the basis at zero level in all phase II computations, there is no problem. However, the problem arises if it becomes positive in some iteration. In such a case, a slightly different approach is adopted in selecting the outgoing variable. The lowest non-negative replacement ratio criterion is not adopted to find the outgoing variable. Artificial variable (or one of the artificial variables if there are more than one) is selected as the outgoing variable. The simplex method can then be applied as usual to obtain the optimal basic feasible solution to the given L.P. problem.

**PHASE II**

Use the optimum basic feasible solution of phase I as a starting solution for the original LPP. Assign the actual costs to the variable in the objective function and a zero cost to every artificial variable in the basis at zero level. Delete the artificial variable column from the table which is eliminated from the basis in phase I. Apply simplex method to the modified simplex table obtained at the end of phase I till an optimum basic feasible is obtained or till there is an indication of unbounded solution.

**REMARKS**

1. In phase I, the iterations are stopped as soon as the value of the new (artificial) objective function becomes zero because this is its minimum value. There is no need to continue till the optimality is reached if this value becomes zero earlier than that.
2. Note that the new objective function is always of minimization type regardless of whether the original problem is of maximization or minimization type.

**EX:**

Consider the following linear programming model and solve it using the two-phase method.
Minimize \( z = 12x_1 + 18x_2 + 15x_3 \)

Subject to
\[
\begin{align*}
4x_1 + 8x_2 + 6x_3 & \geq 64 \\
3x_1 + 6x_2 + 12x_3 & \geq 96 \\
x_1, x_2, x_3 & \geq 0
\end{align*}
\]

Phase 1
The model for phase 1 with its revised objective function is shown below. The corresponding initial table is presented in Table.

Objective function:
\[
\begin{align*}
\text{Min } r &= A_1 + A_2 \\
&= 160 - 7x_1 - 14x_2 - 18x_3 + S_1 + S_2
\end{align*}
\]

subject to
\[
\begin{align*}
4x_1 + 8x_2 + 6x_3 - S_1 + A_1 &= 64 \\
3x_1 + 6x_2 + 12x_3 - S_2 + A_2 &= 64 \\
X_1, X_2, X_3, S_1, S_2, A_1, A_2 &\geq 0
\end{align*}
\]

Initial Table of Phase 1

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>( A_1 )</th>
<th>( A_2 )</th>
<th>( b )</th>
<th>B.V.</th>
<th>A_1</th>
<th>A_2</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>8</td>
<td>6</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>64</td>
<td>B.V.</td>
<td>A_1</td>
<td>A_2</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>12</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>96</td>
<td>A_1</td>
<td>A_2</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>14</td>
<td>18</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>160</td>
<td>B.V.</td>
<td>r</td>
<td></td>
</tr>
</tbody>
</table>

First iteration simplex tableau

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>( A_1 )</th>
<th>( A_2 )</th>
<th>( b )</th>
<th>B.V.</th>
<th>A_1</th>
<th>A_2</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>8</td>
<td>6</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>64</td>
<td>B.V.</td>
<td>A_1</td>
<td>A_2</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>12</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>96</td>
<td>( \text{Leaving variable}_x )</td>
<td>( \text{Entering variable}_x )</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>14</td>
<td>18</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>160</td>
<td>B.V.</td>
<td>r</td>
<td></td>
</tr>
</tbody>
</table>

Optimal simplex tableau of phase 1
The set of basic variables in the optimal table of phase 1 does not contain artificial variables. So, the given problem has a feasible solution.

**Phase 2**
The only one iteration of phase 2 is shown in Table one can verify that Table gives the optimal solution. The solution in Table is optimal and feasible. The optimal results are presented below.

by $x_1=0$, $x_2 =3.2=6/5$, $x_3 = 6.4$ and $\min z=153.6$

**Initial Table of Phase 2**

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$b$</th>
<th>B.V.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>1</td>
<td>0</td>
<td>-0.2</td>
<td>0.1</td>
<td>-0.1</td>
<td>-0.1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0.1</td>
<td>-0.13</td>
<td>0.13</td>
<td>6.4</td>
</tr>
<tr>
<td>-12</td>
<td>-15</td>
<td>-18</td>
<td>0</td>
<td>2.1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Optimal Table of Phase 2**

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$b$</th>
<th>B.V.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>1</td>
<td>0</td>
<td>-0.2</td>
<td>0.1</td>
<td>3.2</td>
<td>$x_2$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0.1</td>
<td>-0.13</td>
<td>6.4</td>
<td>$x_3$</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2.1</td>
<td>153.6</td>
<td>$z$</td>
</tr>
</tbody>
</table>

The optimal results are presented by $x_1=0$, $x_2 =3.2=6/5$, $x_3 = 6.4$ and $\min z=153.6$
UNIT – II TRANSPORTATION PROBLEM :

Introduction:

The Transportation problem is one of the sub classes of L.P.Ps in which the objective is to transport various quantities of a single homogeneous commodity, that are initially stored at various origins to different destinations in such a way that the total transport cost is minimum. To Achieve this objective we must know the amount and location of available supplies and the quantities demanded, In addition, we must know that costs that result from transporting one unit of commodity from various origins to various destinations.

General Transportation Problem:

Let
\[ a_i = \text{quantity of commodity available at origin } i \]
\[ b_j = \text{quantity of commodity needed at destination } j \]
\[ c_{ij} = \text{cost of transporting one unit of commodity from origin } i \text{ to destination } j \]
\[ x_{ij} = \text{quantity transported from origin } i \text{ to destination } j \]

Then the problem is determined the transportation schedule so as to minimize the total transportation cost satisfying supply and demand constraints. Mathematically, the problem may be stated as a LPP as follows.

Minimize \( z = \sum_{i=0}^{m} \sum_{j=0}^{n} x_{ij} c_{ij} \)
subject to the constraints
\[ \sum_{j=0}^{n} x_{ij} = b_j \]
\[ j = 1, 2, \ldots, n \]
\[ \sum_{i=0}^{m} x_{ij} = a_i \]
\[ i = 1, 2, \ldots, n \text{ and } x_{ij} \geq 0, \text{ for all } i \text{ and } j \]

Existence of feasible solution:

A necessary and sufficient condition for existence of a feasible solution to the general transportation problem is
\[ \sum_{i=1}^{m} a_{ij} = \sum_{j=1}^{n} b_{j} = 0 \]

Existence of optimum solution:
There always exists an optimum solution to a transportation problem

Basic feasible solution:

The number of basic variables of the general transportation problem at any stage of feasible solution must be m+n-1.

NOTE:

1. When the total demand is equal to total supply then the transportation table is said to be balanced, otherwise unbalanced.
2. The allocated cells in the transportation table will be called occupied cells and empty cells are called non-occupied cells.

The transportation table:

Since the transportation problem is just a special case of general LPP.

<table>
<thead>
<tr>
<th>Destination</th>
<th>supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>x_{11}</td>
<td>a_1</td>
</tr>
<tr>
<td>x_{12}</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td></td>
</tr>
<tr>
<td>x_{1n}</td>
<td></td>
</tr>
<tr>
<td>X_{21}</td>
<td>a_2</td>
</tr>
<tr>
<td>X_{22}</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td></td>
</tr>
<tr>
<td>X_{2n}</td>
<td></td>
</tr>
<tr>
<td>X_{m1}</td>
<td>am</td>
</tr>
<tr>
<td>X_{m2}</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td></td>
</tr>
<tr>
<td>X_{mn}</td>
<td></td>
</tr>
<tr>
<td>b_1</td>
<td>b_n</td>
</tr>
<tr>
<td>b_2</td>
<td></td>
</tr>
</tbody>
</table>

Demand

Solution of the transportation problem

The solution of a transportation problem involves the following major steps.

Step 1: Formulate the given problem as a LPP.

Step 2: Setup the given LPP in the tabular form known as a Transportation table.

Step 3: Find an initial basic feasible solution that must satisfy all the supply and demand conditions.

Step 4: Examine the solution obtained in step 3 for optimality.

Step 5: If the solution is not optimum modified the shipping schedule by including that unoccupied cell whose inclusion may result in an improved solution.

Step 6: Repeat Step 3 until no further improvement is possible.

We shall now discuss various methods available for finding an initial basic feasible solution and then attaining an optimum solution.
Finding an Initial Basic Feasible Solution:
There are several methods available to obtain an initial basic feasible solution however we shall discuss here the following three methods.

1. North – West Corner Method
2. Least Cost Method
3. Vogles Approximation Method

North – West Corner Method:

Step1: Select the upper left (north-west) cell of the transportation matrix and allocate the maximum possible value to X11 which is equal to min(a1,b1).

Step2:  
• If allocation made is equal to the supply available at the first source (a1 in first row), then move vertically down to the cell (2,1).
• If allocation made is equal to demand of the first destination (b1 in first column), then move horizontally to the cell (1,2).
• If a1=b1 , then allocate X11= a1 or b1 and move to cell (2,2).

Step3: Continue the process until an allocation is made in the south-east corner cell of the transportation table.

Example: Solve the Transportation Table to find Initial Basic Feasible Solution using North-WestCornerMethod.

<table>
<thead>
<tr>
<th></th>
<th>D1</th>
<th>D2</th>
<th>D3</th>
<th>D4</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>19</td>
<td>30</td>
<td>50</td>
<td>10</td>
<td>7</td>
</tr>
<tr>
<td>S2</td>
<td>70</td>
<td>30</td>
<td>40</td>
<td>60</td>
<td>9</td>
</tr>
<tr>
<td>S3</td>
<td>40</td>
<td>8</td>
<td>70</td>
<td>20</td>
<td>18</td>
</tr>
<tr>
<td>Demand</td>
<td>5</td>
<td>8</td>
<td>7</td>
<td>14</td>
<td>34</td>
</tr>
</tbody>
</table>

Least Cost Method:

Step1: Select the cell having lowest unit cost in the entire table and allocate the minimum of supply or demand values in that cell.

Step2: Then eliminate the row or column in which supply or demand is exhausted. If both the supply and demand values are same, either of the row or column can be eliminated.

In case, the smallest unit cost is not unique, then select the cell where maximum allocation can be made.
Step 3: Repeat the process with next lowest unit cost and continue until the entire available supply at various sources and demand at various destinations is satisfied.

The total transportation cost obtained by this method

\[ = 8 \times 8 + 10 \times 7 + 20 \times 7 + 40 \times 7 + 70 \times 2 + 40 \times 3 \]

\[ = Rs. 814 \]

Here, we can see that the \textit{Least Cost Method} involves a lower cost than the \textit{North-West Corner Method}.

\textbf{Vogel's Approximation Method}

This method also takes costs into account in allocation. Five steps are involved in applying this heuristic:

\textbf{Step 1:} Determine the difference between the lowest two cells in all rows and columns, including dummies.

\textbf{Step 2:} Identify the row or column with the largest difference. Ties may be broken arbitrarily.

\textbf{Step 3:} Allocate as much as possible to the lowest-cost cell in the row or column with the highest difference. If two or more differences are equal, allocate as much as possible to the lowest-cost cell in these rows or columns.

\textbf{Step 4:} Stop the process if all row and column requirements are met. If not, go to the next step.
Step 5: Recalculate the differences between the two lowest cells remaining in all rows and columns. Any row and column with zero supply or demand should not be used in calculating further differences. Then go to Step 2.

The Vogel's approximation method (VAM) usually produces an optimal or near-optimal starting solution. One study found that VAM yields an optimum solution in 80 percent of the sample problems tested.

The total transportation cost obtained by this method
\[
= 8*8+19*5+20*10+10*2+40*7+60*2
= Rs.779
\]

Here, we can see that Vogel’s Approximation Method involves the lowest cost than North-West Corner Method and Least Cost Method and hence is the most preferred method of finding initial basic feasible solution.

Assignment problem
- Assignment problem is a particular class of transportation linear programming problems
- Supplies and demands will be integers (often 1)
- Traveling salesman problem is a special type of assignment problem

Objectives
- To structure and formulate a basic assignment problem
To demonstrate the formulation and solution with a numerical example

To formulate and solve traveling salesman problem as an assignment problem

**Structure of Assignment Problem**
- Assignment problem is a special type of transportation problem in which
  - Number of supply and demand nodes are equal.
  - Supply from every supply node is one.
  - Every demand node has a demand for one.
  - Solution is required to be all integers.

**Goal of an general assignment problem:** Find an optimal assignment of machines (laborers) to jobs without assigning an agent more than once and ensuring that all jobs are completed
- The objective might be to minimize the total time to complete a set of jobs, or to maximize skill ratings, maximize the total satisfaction of the group or to minimize the cost of the assignments
- This is subjected to the following requirements:
  - Each machine is assigned no more than one job.
  - Each job is assigned to exactly one machine.

**Formulation of Assignment Problem**
- Consider \( m \) laborers to whom \( n \) tasks are assigned
- No laborer can either sit idle or do more than one task
- Every pair of person and assigned work has a rating
- Rating may be cost, satisfaction, penalty involved or time taken to finish the job
- \( N^2 \) such combinations of persons and jobs assigned
- Optimization problem: Find such job-man combinations that optimize the sum of ratings among all.
- Let \( x_{ij} \) be the decision variable
- The objective function is

\[
\text{Minimize } \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij}
\]

Since each task is assigned to exactly one laborer and each laborer is assigned only one job, the constraints are

\[
\sum_{i=1}^{n} x_{ij} = 1 \quad \text{for } j = 1, 2, ..., n
\]

\[
\sum_{j=1}^{n} x_{ij} = 1 \quad \text{for } i = 1, 2, ..., m
\]

\[
x_{ij} = 0 \text{ or } 1
\]

- Due to the special structure of the assignment problem, the solution can be found out using a more convenient method called Hungarian method.
Unit – III Replacement
Replacement

The study of replacement is concerned with situations that arise when some items such as machines, men, electric bulbs etc., need to replace due to their deteriorating efficiency, failure, or break-down. The deteriorating efficiency or break-down may be either gradual or all of a sudden.

Following are the situations when the replacement of certain items needs to be done:

1. An old item has failed and does not work at all, or the old item is expected to fail shortly.
2. The old item has deteriorated and works badly or requires expensive maintenance.
3. A better design of equipment has been developed.

Replacement problems can be broadly classified into two types:

1. When the equipment/asset deteriorates with time and the value of money:
   a) Does not change with time.
   b) Changes with time.
2. When items/units fail completely all of a sudden.

Replacement of Items that Deteriorate Gradually:

Generally, the cost of maintenance and repair of certain items (equipments) increases with time and a stage may come when these costs become so high that it is more economical to replace the item by a new one.

At this point, a replacement is justified.

Replacement policy when value of money does not change with time:

The aim here is to determine the optimum replacement age of an item whose maintenance cost increases...
With time and the value of money remains static during that period.

Let 
\[ C : \text{capital cost of equipment/item.} \]
\[ S : \text{scrap value of the item.} \]
\[ n : \text{no. of years that item would be in use.} \]
\[ f(t) : \text{maintenance cost functions.} \]
\[ A(n) : \text{average total annual cost.} \]

**Case 1:**

When \( t \) is a continuous variable:

If the equipment is used for \( n \) years, then the total cost incurred during this period is

\[ T_C = \text{capital cost} - \text{scrap value} + \text{maintenance cost} \]

\[ = C - S + \int_0^n f(t) \, dt \]

Average annual total cost is

\[ A(n) = \frac{1}{n} T_C = \frac{C - S}{n} + \frac{1}{n} \int_0^n f(t) \, dt \]

For minimum cost,

Differentiate \( A(n) \) with \( n \) and equate to zero

\[ \frac{d}{dn} A(n) = 0 \]

\[ = \frac{(C-S)}{n^2} - \frac{1}{n} \int_0^n f(t) \, dt + \frac{1}{n} f(n) = 0 \]

\[ f(n) = \frac{C-S}{n} + \frac{1}{n} \int_0^n f(t) \, dt \approx A(n) \]
Case 2: When \( t \) is a discrete variable

Here, the period of time is considered as fixed and \( n, t \) take values \( 1, 2, 3, \ldots \), then

\[
A(n) = \frac{C_S + \sum_{t=1}^{n} E_s(t)}{n}
\]

Now \( A(n) \) will be minimum for that value of \( n \), for which

\[ A(n+1) > A(n) \quad \text{and} \quad A(n-1) > A(n) \]

Or

\[ A(n+1) - A(n) > 0 \quad \text{and} \quad A(n) - A(n-1) > 0 \]

\[
A(n+1) = \frac{C_S + \sum_{t=1}^{n+1} E_s(t)}{n+1}
\]

\[
= \frac{1}{n+1} \left[ C_S + \sum_{t=1}^{n} E_s(t) + E_s(n+1) \right]
\]

\[
= \frac{1}{n+1} \left[ nA(n) + E_s(n+1) \right]
\]

\[ \Rightarrow A(n+1) - A(n) = \frac{1}{n+1} \left[ -E_s(n+1) - A(n) \right] \]

\[ \Rightarrow A(n) - A(n-1) > 0 \]

\[ \Rightarrow A(n) \leq A(n-1) \]

Similarly, \( A(n-1) \geq A(n) \)

This suggests the optimal replacement policy:

Replace the equipment at the end of \( n \) years, if the maintenance cost in the \((n+1)th\) year is more than the average total cost in the \(n\)th year and \((n+1)\)th year maintenance cost is less than the previous year's average total cost.
Procedure:

The procedure for obtaining the decision when to replace the equipment in the case of money value not changing with time can be obtained as follows:

1. Draw the table

<table>
<thead>
<tr>
<th>Years of Service</th>
<th>Cost of Item</th>
<th>Running Cost</th>
<th>Cumulative Value</th>
<th>Salvage Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(n)</td>
<td>(c)</td>
<td>(c)</td>
<td>(c+c)</td>
<td>$400</td>
</tr>
</tbody>
</table>

Total Cost: $400
Replacement policy for items when money value changes with time

When the time value of money is taken into consideration
We shall assume that
(i) The equipment in question has no scrap value
(ii) The maintenance costs are incurred in the
beginning of the different time periods.

It is assumed that the maintenance cost increases with
year and each cost is to be paid just in the start of the period.
Let the money, carry a rate of interest \(r\) per year. Thus a
rupee invested now will be worth \((1+r)\) after a year, \((1+r)^2\) after
two years. Similarly, a rupee invested today will be worth \((1+r)^n\),
n years hence, or, in other words, if we have to make a payment
of one rupee in \(n\) years time, it is equivalent to making a
payment of \((1+r)^n\) rupees today. The quantity \((1+r)^n\) is called
the present worth factor of one rupee spent in \(n\) years time
from now onwards. The expression \((1+r)^n\) is known as the payment
compound amount factor.

Let the initial cost of the equipment be \(C\).

Let \(R_n\) be the operating cost in year \(n\).

Let \(r\) be the rate of interest in such a way that
\(r = (1+r)^{-1}\), is the discount rate. (or) discount cost.

The present value of all future discounted costs \(V_n\) associated
with a policy of replacing the equipment at the end of each
\(n\) years is given by:

\[
V_n = \left\{ (C+R_0) + r R_1 + r^2 R_2 + \cdots + r^{n-1} R_{n-1} \right\} + \left\{ (C+R_0) r + r^2 R_1 + \right.
\left. r^3 R_2 + \cdots + r^{n-1} R_{n-1} \right\} + \cdots
\]

Now, \(V_n\) will be a minimum for that value of \(n\) for which
\(V_{n+1} - V_n > 0 \quad \text{and} \quad V_{n+2} - V_{n+1} < 0\).

\(\Rightarrow R_n > (1+r) V_n\)
\(\Rightarrow R_{n+1} < (1+r) V_n\)
\[ R_{n-1} < \frac{C + R_0 + vR_1 + v^2R_2 + \cdots + v^nR_{n-1}}{1 + v + v^2 + \cdots + v^n} \leq R_n. \]

The expression which lies below \( R_{n-1} \) and \( R_n \) is called "weighted average cost" of all years, with weights \( 1, v, v^2, \ldots, v^n \) respectively.

Hence, the optimal replacement policy of the equipment after \( n \) years is:

(a) Do not replace the equipments if the next period's operating cost is less than the weighted average of previous costs.

(b) Replace the equipments if the next period's operating cost is greater than the weighted average of previous costs.

Selection of the best equipment amongst two:

Procedure for the selection of an economically best item amongst the available equipments:

1. Considering the case of two equipments, let A and B, first we have to find the best replacement age for both the equipments by using:

\[ R_{n-1} < (1-v) V_n < R_n. \]

Let the optimum replacement age for A and B comes out be \( n_1 \) and \( n_2 \) respectively.

2. Compute the fixed annual payment (or weighted average cost) for each item by using the formula

\[ W(n) = \frac{C + R_0 + vR_1 + v^2R_2 + \cdots + v^nR_{n-1}}{1 + v + v^2 + \cdots + v^n}. \]

and substitute \( n = n_1 \) for equipment A

\[ n = n_2 \] for equipment B.
(3) If \( W(n) < W(n_2) \), choose equipment A.
(4) If \( W(n) > W(n_2) \), choose equipment B.
(5) If \( W(n) = W(n_2) \), both equipments are equally good.

Steps to find the policy when money value changes with time:

Step 1: Note the values of capital cost of machine, salvage value, rate of depreciation, PUF etc.

Construction of the Tabular Form

<table>
<thead>
<tr>
<th>Years</th>
<th>Running cost</th>
<th>PUF</th>
<th>Rate of depreciation</th>
<th>Total (Net) Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(P_0)</td>
<td></td>
<td>(r)</td>
<td>(P_0)</td>
</tr>
<tr>
<td>1</td>
<td></td>
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<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\( e^{-r_1} \)  \( e^{-r_2} \) \( e^{-r_3} \) \( e^{-r_4} \) \( e^{-r_5} \)
Group Replacement:

Group Replacement of items that fail completely:

This policy is concerned with the items that either work perfectly, or work partially, or inefficiently or fail completely. This situation generally happens when the system consists of a large number of identical low cost items that are increasingly liable to failure with age. In such cases, the replacement of individual items would incur a set of costs, which is independent of the number replaced. However, it may be advantageous to replace all the items at a time at a fixed interval. This policy is known as group replacement policy and is very attractive, particularly when:

1) The value of any individual item is so small.
2) The cost of keeping records of individual ages is high that cannot be justified.
3) The purchase of such identical items in bulk can be had at discounted rate.
4) Average individual replacement would be costlier than the average group replacement.
5) If sufficient number of standby machines are available.
6) New designs of the equipment considerably increase the production rate.

In all the above cases, two types of replacement policies considered are:

1) Individual Replacement:
Under this policy, an item is replaced immediately after its failure.

2) Group Replacement:
Under this policy, a decision will be taken so as to replace all the items irrespective of the fact that the items have failed - not failed, provided if any item fails, before the optimal time it may be replaced "in time" individually.
Unit –IV Game Theory

Introduction

Game theory is a formal methodology and a set of techniques to study the interaction of rational agents in strategic settings. ‘Rational’ here means the standard thing in economics: maximizing over well-defined objectives; ‘strategic’ means that agents care not only about their own actions, but also about the actions taken by other agents. Note that decision theory — which you should have seen at least a bit of last term — is the study of how an individual makes decisions in non-strategic settings; hence game theory is sometimes also referred to as multi-person decision theory. The common terminology for the field comes from its putative applications to games such as poker, chess, etc. However, the applications we are usually interested in have little directly to do with such games. In particular, these are what we call “zero-sum” games in the sense that one player’s loss is another player’s gain; they are games of pure conflict. In economic applications, there is typically a mixture of conflict and cooperation motives.

What is a Game?

A game is a formal description of a strategic situation.

Game theory

Game theory is the formal study of decision-making where several players must make choices that potentially affect the interests of the other players.

Elements of a Game

There are three main elements of a game:

- The players.
- The strategies of each player.
- The consequences (payoffs) for each player for every possible profile of strategy choices of all players.

Fundamental Principles of Game Theory

When analyzing any game, we make the following assumptions about both players:

- Each player makes the best possible move.
- Each player knows that his or her opponent is also making the best possible move.

TYPES OF GAMES

1. Two-person games and n-person games.

In two person games, the players may have many possible choices open to them for each play of the game but the number of player’s remains only two. Hence it is called a two person game. In case of more than two persons, the game is generally called n-person game.
2. Zero sum game.
A zero sum game is one in which the sum of the payment to all the competitors is zero for every possible outcome of the game in a game if the sum of the points won equals the sum of the points lost.

3. Two person zero sum game
A game with two players, where the gain of one player equals the loss to the other is known as a two person zero sum game. It also called rectangular form. The characteristics of such a game are.
(i) Only two players participate in the game.
(ii) Each player has a finite number of strategies to use.
(iii) Each specific strategy results in a payoff.
(iv) Total payoff to the two players at the end of each play is zero.

SADDLE POINT
A saddle point is a position in payoff matrix where the maximum of row minima coincides with the minimum of column maxima. The payoff at the saddle point is called the value of the game.

Strategy:
A Strategy is a set of best choices for a player for an entire game.

Pure Strategy:
A pure strategy defines a specific move or action that a player will follow in every possible attainable situation in a game.

Mixed strategy:
A mixed strategy is an active randomization, with given probabilities that determines the player’s decision. As a special case, a mixed strategy can be the deterministic choice of one of the given pure strategies.

Payoff :
The payoff or outcome is the state of the game at its conclusion. In games like chess the payoff is win or loss.

Payoff matrix
Suppose the player A has ‘m’ activities and the player B has ‘n’ activities. Then a payoff matrix can be formed by adopting the following rules
1. Row designations for each matrix are the activities available to player A
2. Column designations for each matrix are the activities available to player B
3. Cell entry Vij is the payment to player A in A’s payoff matrix when A chooses the activity i and B chooses the activity j.
4. With a zero-sum, two-person game, the cell entry in the player B’s payoff matrix will be negative of the corresponding cell entry Vij in the player A’s payoff matrix so that sum of payoff matrices for player A and player B is ultimately zero.
Value of the game

Value of the game is the maximum guaranteed game to player A (maximizing player) if both the players use their best strategies. It is generally denoted by ‘V’ and it is unique.

THE MAXIMIN-MIINMAX PRINCIPLE

This principle is used for the selection of optimal strategies by two players. Consider two players A and B. A is a player who wishes to maximize his gain while player B wishes to minimize his losses. Since A would like to maximize his minimum gain, we obtain for player A, the value called maximize value and the corresponding strategy is called maximize strategy. On the other hand, since player B wishes to minimize his losses, a value called the Minimax value which is the minimum of the maximum losses is found. The corresponding strategy is called the minimax strategy. When these two are equal, the corresponding strategies are called optimal strategy and the game is said to have a saddle point. The value of the game is given by the saddle point. The selection of maximin and minimax strategies by A and B is based upon the so-called maximin–minimax principle which guarantees the best of the worst results.

Dominance property

Sometimes it is observed that one of the pure strategies of either player is always inferior to at least one of the remaining ones. The superior strategies are said to dominate the inferior ones. In such cases of dominance, we reduce the size of the payoff matrix by deleting those strategies which are dominated by others. The general rules for dominance are:

I. If all the elements of a row are less than or equal to the corresponding elements of any other row then row is dominated by the row.

II. If all the elements of a column are greater than or equal to the corresponding elements of other column then column, then column is dominated by the column.

III. Dominated rows and columns may be deleted to reduce the size of the pay-off matrix as the optimal strategies will remain unaffected.

IV. If some linear combinations of some rows dominates row, then the row will be deleted. Similar arguments follow for column.
UNIT – V WAITING LINES

Queueing theory is the mathematical study of waiting lines, or queues.

Significance:

We come in contact with waiting line systems, or queuing systems, everywhere and everyday. May it be waiting in line for your morning coffee, opening up your email account to see your list of new messages, or stopping at a red light in a traffic intersection, you are participating in a waiting lines system.

As a business, waiting line systems are especially important to the operations management of the organization.

Structure of waiting lines:

Each specific situation will be different, but waiting line systems are essentially composed of four major elements:

1. An input of customers or items to make a customer population. This population can be finite or infinite
2. A waiting line system of customers or items
3. Workstations or operations that perform one or more activities
4. A priority rule that selects the next customer or item on which the activities perform

SINGLE-SERVER WAITING LINE MODEL

The easiest waiting line model involves a single-server, single-line, single-phase system.

The following assumptions are made when we model this environment:
1. The customers are patient (no balking, reneging, or jockeying) and come from a population that can be considered infinite.
2. Customer arrivals are described by a Poisson distribution with a mean arrival rate of $\lambda$ (lambda). This means that the time between successive customer arrivals follows an exponential distribution with an average of $1/\lambda$.
3. The customer service rate is described by a Poisson distribution with a mean service rate of $\mu$ (mu). This means that the service time for one customer follow an exponential distribution with an average of $1/\mu$.
4. The waiting line priority rule used is first-come, first-served.

Using these assumptions, we can calculate the operating characteristics of a waiting line system using the following formulas:
MULTISERVER WAITING LINE MODEL

In the single-line, multi-server, single-phase model, customers form a single line and are served by the first server available. The model assumes that there are \( s \) identical servers, the service time distribution for each server is exponential, and the mean service time is \( 1/\mu \). Using these assumptions, we can describe the operating characteristics with the following formulas:

\( s \) = the number of servers in the system

\( p = \frac{\lambda}{s \mu} \) = the average utilization of the system

\( P_0 = \left[ \sum_{n=0}^{s-1} \frac{(\lambda/\mu)^n}{n!} \right] + \frac{(\lambda/\mu)^s}{s!} \left( \frac{1}{1-p} \right)^{-1} \) = the probability that no customers are in the system

\( L_s = \frac{P_s(\lambda/\mu)p}{s!(1-p)^2} \) = the average number of customers waiting in line

\( W_s = \frac{L_s}{\lambda} \) = the average time spent waiting in line

\( W = \frac{W_s}{\mu} = \) the average time spent in the system, including service

\( L = \lambda W \) = the average number of customers in the system

\( P_n = \begin{cases} (\lambda/\mu)^n \frac{P_0}{n!} & \text{for } n \leq s \\ (\lambda/\mu)^n \frac{P_0}{s!} \frac{1}{s^{s-n}} & \text{for } n > s \end{cases} \) = the probability that \( n \) customers are in the system at a given time

Distribution Of Arrivals:

When describing a waiting system, we need to define the manner in which customers or the waiting units are arranged for service. Waiting line formulas generally require an arrival rate, or the number of units per period (such as an average of one every six minutes). A constant arrival distribution is periodic, with exactly the same time between successive arrivals. In productive systems, the only arrivals that truly approach a constant interval period are those subject to machine control. Much more common are variable (random) arrival distributions. In observing arrivals at a service facility, we can look at them from two viewpoints: First, we can analyze the time between successive arrivals to see if the times follow some statistical distribution. Usually
we assume that the time between arrivals is exponentially distributed. Second, we can set some time length \((T)\) and try to determine how many arrivals might enter the system within \(T\). We typically assume that the number of arrivals per time unit is Poisson distributed.

**Exponential Distribution**
In the first case, when arrivals at a service facility occur in a purely random fashion, a plot of the inter arrival times yields an exponential distribution. The probability function is

\[
f(t) = \lambda e^{-\lambda t}
\]

where \(\lambda\) is the mean number of arrivals per time period.

**Poisson Distribution**

In the second case, where one is interested in the number of arrivals during some time period \(T\), the distribution obtained by finding the probability of exactly \(n\) arrivals during \(T\). If the arrival process is random, the distribution is the Poisson, and the formula is

\[
P_T(n) = \frac{(\lambda T)^n e^{-\lambda T}}{n!}
\]
UNIT – VI INVENTORY
20. 1. INTRODUCTION
In our daily life, we observe that a small retailer knows roughly the demand of his customers in a month or a week, and accordingly places orders on the wholesaler to meet the demand of his customers. But, this is not the case with a manager of a big departmental store or a big retailer, because the stocking in such cases depends upon various factors, e.g. demand, time of ordering, lag between orders and actual receipts, etc. So the real problem is to have a compromise between over-stocking and under-stocking.

The study of such type of problems is known by the term ‘Material Management’ or ‘Inventory Control’. The inventory control may be defined as follows:

Definition. The function of directing the movement of goods through the entire manufacturing cycle from the requisitioning of raw materials to the inventory of finishes goods orderly mannered to meet the objectives of maximum customer-service with minimum investment and efficient (low-cost) plant operation.

The models discussed in this chapter will be limited mainly to the elementary type, because the analytical study of the other cases becomes more difficult. After a general discussion of each indicated type of model, we shall give many interesting solved examples so that all the necessary ideas may be clear to the students. We shall also discuss another class of inventory models, namely ‘Inventory Models with Price Breaks’ (i.e. Quantity Discount Models).

20.2 WHAT IS INVENTORY?
In broad sense, inventory may be defined as the stock of goods, commodities or other economic resources that are stored or reserved in order to ensure smooth and efficient running of business affairs.

The inventory or stock of goods may be kept in any of the following forms:

(i) Raw material inventory, i.e. raw materials which are kept in stock for using in the production of goods.

(ii) Work-in-process inventory, i.e. semi finished goods or goods in process which are stored during the production process.

(iii) Finished goods inventory, i.e. Finished goods awaiting shipment from the factory.
(iv) **Inventory also include: furniture, machinery, fixtures, etc.**

(v) i.e. semi finished goods or goods in process which are stored during the production process.

The term inventory may be classified in two main categories.

1 – Direct Inventories

The items which play a direct role in the manufacture and become an integral part of finished goods are included in the category of direct inventories. These may be further classified into four main groups:

(a) **Raw material inventories** are provided:
   (i) for economical bulk purchasing, (ii) to enable production rate changes (iii) to provide production buffer against delays in transportation, (iv) for seasonal fluctuations.

(b) **Work-in-process inventories** are provided:
   (i) To enable economical lot production, (ii) to cater to the variety of products (iii) for replacement of wastages, (iv) to maintain uniform production even if amount of sales may vary.

(c) **Finished-goods inventories** are provided:
   (i) For maintaining off-self delivery, (ii) to allow stabilization of the production level (iii) for sales promotion.

(d) **Spare parts.**

2 – Indirect Inventories

Indirect inventories include those items which are necessarily required for manufacturing but do not become the component of finished production, like: oil, grease, lubricants petrol, office-material, maintenance material, etc.
20.5. HOW TO DEVELOP AN INVENTORY MODEL?

As explained earlier, inventory models are concerned with two main decisions: how much to order at a time and when to order so as to minimize the total cost. The sequence of basic steps required for developing an inventory model may be organized as follows:

Step 1. First take the physical stock of all the inventory items in an organization.

Step 2. Then, classify the stock of items into various categories. Although several methods are available to classify the inventories, but the selected method must serve the objectives of inventory management. For example, inventory items may be classified as raw materials, work-in-process, purchased components, consumable stores and maintenance spare, and finished goods, etc.

Step 3. Each of above classifications may be further divided into several groups. For example, consumable stores and maintenance spares can be further divided into the following groups: (i) building materials, (ii) hardware items, (iii) lubricants and oils, (iv) textiles and fibres, (v) electric spares, (vi) mechanical spares, (vii) stationary items, etc.

Step 4. After classification of inventories, each item should be assigned a suitable code. Coding system should be flexible so that new items may also be permitted for inclusion.

Step 5. Since the number of items in an organization is very large, separate inventory management model should be developed for each category of items.

Step 6. Use A-B-C of V-E-D classification (as discussed in the next chapter) which provide a basis for a selective control of inventories through formulation of suitable inventory policies for each category.

Step 7. Now decide about the inventory model to be developed. For example, fixed-order-quantity system may be developed for 'A' class and high valued 'B' class items, whereas periodic review system may be developed for low valued 'B' class and 'C' class items.

Step 8. For this, collect data relevant to determine ordering cost, shortage cost, inventory carrying cost, etc.

Step 9. Then, make an estimate of annual demand for each inventory item and their prevailing market price.

Step 10. Estimate lead-time, safety stock and reorder level, if supply is not instantaneous. Also decide about the service-level to be provided to the customers.

Step 11. Now develop the inventory model.

Step 12. Finally, review the position and make suitable alterations, if required, due to current situations or constraints.
(ii) **Anticipation Inventories.** These are built up in advance for the season of large sales, a promotion programme or a plant shut-down period. In fact, anticipation inventories store the men and machine hours for future requirements.

(iii) **Cycle(lot-size) Inventories.** In practical situations, it seldom happens that the rate of consumption is the same as the rate of production or purchasing. So the items are procured in larger quantities that they are required. This results in cycle (or lot-size) inventories.

(iv) **Transportation Inventories.** Such inventories exist because the materials are required to move from one place to another. When the transportation time is long, the items under transport cannot be served to customers. These inventories exist solely because of transportation time.

(v) **Decoupling Inventories.** Such inventories are needed for meeting out the demands during the decoupling period of manufacturing or purchasing.

### 20.4. INVENTORY DECISIONS

The managers must take two basic decisions in order to accomplish the functions of inventory. The decisions made for every item in the inventory are:

1. **How much amount of an item should be ordered when the inventory of that item is to be replenished?**

2. **When to replenish the inventory of that item?**

Inventory decisions may be classified as follows:

```
INVENTORY DECISIONS

How much to order?  When to order?

- Demand
  - Deterministic
  - Probabilistic

- Supply
  - Fixed order system
  - Fixed period system

- Rate of supply
  - Deterministic
  - Probabilistic

- Lead time
```

Before taking inventory decisions, it is necessary to develop an inventory model.
Before we proceed to discuss inventory models, it is very desirable to consider briefly the costs involved in the inventory decisions.

**20.6. COSTS INVOLVED IN INVENTORY PROBLEMS**

1. **Holding Cost (C₁ or Cₖ).** The cost associated with carrying or holding the goods in stock is known as *holding or carrying* cost which is usually denoted by C₁ or Cₖ per unit of goods for a unit of time. Holding cost is assumed to vary directly with the size of inventory as well as the time for which the item is held in stock. The following components constitute the holding cost:

   (i) **Invested Capital Cost.** This is the interest charge over the capital investment. Since this is the most important component, a careful investigation is required to determine its rate.

   (ii) **Record-Keeping and Administrative Cost.** This signifies the need of keeping funds for maintaining the records and necessary administration.

   (iii) **Handling Costs.** These include all costs associated with movement of stock such as:

           Cost of labour, over-head cranes. Gantry and other machinery required for this purpose.

   (iv) **Storage Costs.** These involve the rent of storage space or depreciation and interest even if the own space is used.

   (v) **Depreciation, Deterioration and Obsolescence Costs.** Such costs arise due to the items in stock being out of fashion or the items undergoing chemical changes during storage (e.g. rusting in steel).

   (vi) **Taxes and Insurance Costs.** All these costs require careful study and generally amounts to 1% to 2% of the invested capital.

   (vii) **Purchase Price or Production Costs.** Purchase price per-unit item is affected by the quantity purchased due to quantity discounts or price-breaks. Production cost per unit item depends upon the length of production runs. For long smooth production runs this cost in lower due to more efficiency of men and machines. So the order quantity must be suitably modified to take the advantage of these price discounts.

       If \( P \) is the purchase price of an item and \( h \) is the stock holding cost per unit item expressed as a fraction of stock value (in rupees), then the holding cost \( C₁ = IP \).

   (viii) **Salvage Costs or Selling Price.** When the demand for an item is affected by its quantity in stock, the decision model of the problem depends upon the profit maximization criterion and includes the revenue (sales tax etc.) from the sale of the item. Generally, salvage costs are combined with the storage costs and not considered independently.
2. Shortage Costs or Stock-out Costs ($C_s$ or $C_o$). The penalty costs that are incurred as a result of running out of stock (i.e., shortage) are known as shortage or stock-out costs. These are denoted by $C_s$ or $C_o$ per unit of goods for a specified period.

These costs arise due to shortage of goods, sales may be lost, goodwill be lost either by a delay in meeting the demand or being quite unable to meet the demand at all. In the case where the unfilled demand for the goods can be satisfied at a later date (backlog case), these costs are usually assumed to vary directly with the shortage quantity and the delaying time both. On the other hand, if the unfilled demand is lost (no backlog case), shortage costs become proportional to shortage quantity only.

3. Set-up Costs ($C_s$ or $C_o$). These include the fixed cost associated with obtaining goods through placing of an order or purchasing or manufacturing or setting up a machinery before starting production. So, they include costs of purchase, requisition, follow-up, receiving the goods, quality control, etc. These are also called order costs or replenishment costs, usually denoted by $C_s$ or $C_o$ per production run (cycle). They are assumed to be independent of the quantity ordered or produced.

20. Why Inventory is Maintained?

As we are aware of the fact that the inventory is maintained for efficient and smooth running of business affairs. If a manufacturer has no stock of goods at all, on receiving a sale-order he has to place an order for purchase of raw materials, wait for their receipt and then start his production. Thus, the customers will have to wait for a long time for the delivery of the goods and may turn to other suppliers. This results in a heavy loss of business. So it becomes necessary to maintain an inventory because of the following reasons:

(i) Inventory helps in smooth and efficient running of business.
(ii) Inventory provides service to the customers immediately or at a short notice.
(iii) Due to absence of stock, the company may have to pay high prices because of piece-wise purchasing. Maintaining of inventory may earn price discount because of bulk purchasing.
(iv) Inventory also acts as a buffer stock when raw materials are received late and so many sale-orders are likely to be rejected.
(v) Inventory also reduces product costs because there is an additional advantage of batching and longer smooth running production runs.
Unit – VII

Dynamic Programming:

Many decision making problems involve a process that takes place in several stages in such a way that at each stage, the process is dependent on the strategy chosen. Such types of problems are called Dynamic Programming Problems. Thus dynamic programming is concerned with the theory of multi stage decision process.

Principle of Optimality

It may be interesting to note that the concept of dynamic programming is largely based upon the principle of optimality due to Bellman, viz.,

“An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must continue an optimal policy with regard to the state resulting from the first decision”

The principle of optimality implies that given the initial state of a system, an optimal policy for the subsequent stage does not depend upon the policy adopted at the preceding stages. That is, the effect of a current policy decision on any of the policy decisions of the preceding stages need not be taken into account at all. It is usually referred to as the Markovian property of dynamic programming.
2.2.2 Bellman’s principle of optimality

Bellman’s celebrated principle of optimality is the key for a constructive approach to the solution of the minimization problem (2.5) which readily leads to a powerful algorithmic tool: the Dynamic Programming algorithm (DP algorithm).

For intermediate states $x_k \in S_k, 1 \leq k \leq N - 1$, that occur with positive probability, we consider the minimization subproblems

\[(2.8a) \min_{\pi_k \in \Pi_k} J_{\pi_k}(x_0) = E \left[ g_N(x_N) + \sum_{\ell=k}^{N-1} g_\ell(x_\ell, \mu_\ell(x_\ell), w_\ell) \right],\]

subject to

\[(2.8b) \ x_{\ell+1} = f_\ell(x_\ell, \mu_\ell(x_\ell), w_\ell), \quad k \leq \ell \leq N - 1,\]

where $\pi_k = \{\mu_k, \mu_{k+1}, \ldots, \mu_{N-1}\}$ and $\Pi_k$ is the set of admissible policies obtained from $\Pi$ by deleting the admissible control laws associated with the previous time instants $0 \leq \ell \leq k - 1$. 
UNIT – VIII SIMULATION
SIMULATION

1. INTRODUCTION:

It is evident that there are many problems of real life which cannot be represented mathematically due to the stochastic nature of the problem, the complexity in problem formulation, or the conflicting ideas needed to properly describe the problem under study. Under such circumstances simulation is often used when all else fail. This method is often viewed as a “method of last resort.”

Simulation analysis is a natural and logical extension to the analytical and mathematical techniques used for solving the problems in Operations Research. Simulation which can appropriately be known as management laboratory, determines the effect of alternate policies without disturbing the real system. Recent advances in simulation methodologies, software availability, and technical developments have made simulation one of the most widely used and popularly accepted tool in ‘operations research’ and ‘systems analysis’. It helps us in deciding the best policy with the prior assurances that its implementation will certainly prove to be beneficial to the organization.

The simulation technique has long been applied by the analysts and designers in physical sciences and it has now become an important tool for dealing with the complicated problems of managerial decision making. The simulated models of aircrafts are tested in wind tunnels to examine their aerodynamic characteristics and the scale models of machines are used to simulate the plant layouts.

The first important application of simulation was probably made by John von Neumann and Stanislaw Ulam for determining the complicated behavior of neutrons in a nuclear shielding problem being too complex for mathematical analysis. After getting the remarkable success of this technique on neutron problem, it became more popular and found many applications in business and industry. In early 1950s, the development of digital computer further increased the rapid progress in the simulation techniques.

Simulation is one of the easiest tools of management science to use, but probably one of the hardest to apply properly and perhaps most difficult from which to draw accurate conclusions. Due to widespread availability of digital computers, simulation becomes
readily available to most of the engineers and managers engaged in operations research projects. Regardless of these drawbacks, simulation is a useful technique and one which is specially suitable for complicated operations research and systems analysis problems.

The purpose of this chapter is to examine the process of simulation and necessary tools to perform such analysis. A special emphasis is given to the Monte-Carlo method of simulation. Some simple examples are discussed to explain the Monte-Carlo technique. To obtain the reliable results the use of computer is very essential in all simulation problems. But for easy demonstration of simulation technique, the numerical examples have been solved by hand computations only.

2. WHAT IS SIMULATION?

In fact, simulation is the representative model for real situations. While visiting some trade-fairs and exhibitions we often find a number of simulated environments therein. For example, a children's cycling park with various signals and crossings in the exhibition is a simulated (represented) model of city-traffic in real system. Also, a simple example is the testing of an aircraft model in a wind tunnel from which the performance of the real aircraft is determined for being fit under real operating conditions. In the laboratories we often perform a number of experiments on simulated models to predict the behavior of the real system under true environments. The environments in a museum of natural history and in a geological garden are also good examples of simulation.

Another idea of simulation is involved in-flight simulators for training pilots. A computer directs the student's handling of the controls in a simulated aeroplane flight deck. The instruments are then operated by the computer to give the same readings which they would in a real flight. An instructor can intervene with 'catastrophes' like an engine failure or a bad storm and a television camera is moved over a model of some country side to give the trainee visual feedback of how the aircraft is behaving.

The combination of computing and simulation has also resulted in the production of TV games. Players interrupt the way a computer program moves various images around the screen from a keyboard or hand-held controller. The computer incorporates their responses into these movements in accordance with the rules of the particular game. Incidentally,
such programs make extensive use of random numbers to find the deflection of tennis balls, the positioning of hostile space ships, etc.

Actually the idea of simulating real system for enjoyment purposes is already known to us. The chess-playing game is a non-probabilistic simulation of a fight between the black and white armies. The game of snake and ladders was initially proposed to simulate the moral progress of the players who moved up ladders when they were 'good' and fell down snakes, indicating temptation, when they were bad. Like in many other board games, dice are used as random number generators.

In all these examples, we have tried to represent the reality to observe-what would happen under real operating situations. Thus, such representation of reality, which may be either in physical form or in a mathematical equations form, may be called simulation.

3. DEFINITIONS OF SIMULATION:

Before we proceed further, it becomes necessary to define the term simulation in more suitable forms. Following few definitions are given below:

Definition 1. Simulation is a representation of reality through the use of a model or other device which will react in the same manner as reality under a given set of conditions.

Definition 2. SIMP, Kolkata Simulation is the use of system model that has designed the characteristics of reality in order to produce the essence of actual operation.

Definition 3. According to Donald G. Malcolm, a simulated model may be defined as one which depicts the working of a large scale system of men, machines, materials and information operating over a period of time in a simulated environment of the actual real world conditions.

Definition 4. According to T.H. Naylor et al. (1966), simulation is a numerical technique for conducting experiments on a digital computer, which involves certain types of mathematical and logical relationships necessary to describe the behaviors and structure of a complex real-world system over extended periods of time.

Definition 5. Churchman has defined simulation as follows:"
"X simulates Y" is true if and only if: (a) X and Y are formal systems; (b) Y is taken to be the real system; (c) X is taken to be an approximation to the real system; and (d) the rules of validity in X are non-error-free, otherwise X will become the real system.

4. Types of simulation:

Simulation is mainly of two types:

(i) Analogue Simulation (or Environmental Simulation). The simple examples cited is Sec. 17.2 are of simulating the reality in physical form, which we may refer as analogue (or environmental) simulation.

(ii) Computer Simulation (or System Simulation). For the complex and intricate problems of managerial decision making, the analogue simulation may not be applicable, and the actual experimentation with the system may be uneconomical also. Under these situations, the complex system is formulated into a mathematical model for which a computer programme is developed, and then the problem is solved by using high speed electronic computer. Such type of simulation is called a computer simulation or system simulation.

The simulation models can be classified into following four categories:

(a) Deterministic models. In these models, input and output variables are not permitted to be random variables and models are described by exact functional relationship.

(b) Stochastic models. In these models, at least one of the variables or functional relationship is given by probability functions.

(c) Static models. These models do not take variable time into consideration.

(d) Dynamic models. These models deal with time varying interaction.

5 Why Simulation is used?

It has been already discussed in the chapter on "What is Operations Research?" that mainly following techniques are adopted for solving various types of managerial decision making problems in operations research.

(i) Scientific method, (ii) Analytical method, and (iii) Iterative method. But each method has its own drawbacks and limitations as discussed below:
8. Event – Type Simulation:

The event type simulation can be understood by discussing the following example.

**Example 1.** Consider a situation where customers arrive at a one-man barber shop for hair cutting. The problem is to analyse the system in order to evaluate the quality of service and the economic feasibility of offering the service. To measure the quality of service one has to make the assessment of the average waiting time per customer and the percentage of time the barber remains idle. Construct the simulation model.

**Solution: Construction of the model.** To construct the model of this system we observe that the changes involved in the analysis of the system can occur only when a customer arrives for service or departs after completion of service. When a customer reaches the barber’s shop, he will have to wait, if the server (barber) is busy. But, on the other hand, a departure of customer after the completion of his service (hair cut) indicates that the server is available to serve the waiting customers, if any one is present. From this, we conclude that only two-types of events can occur, i.e. an arrival event and a departure event. It shows that as the simulator progresses on the time scale, we must pay due attention to one system whenever an event occurs.

**FIG.**

Let us denote the arrival event by \( E_a \), the departure event by \( E_d \), and the simulated period (time span) by \( T \). The simulator starts at time 0 and first progresses up to \( t = t_1 \), then \( t = t_2 \) and so on, until the entire simulated period \( T \) is covered. The above diagram (Fig. 17.1) shows the occurrences of event \( E_a \) and \( E_d \) over the time period \( T \), where the simulation starts by generating \( E_a \) at \( t_1 \). In the beginning, when the service facility is free, the service of the customer will be started immediately. Then the following two new events must be generated:

(i) The next arrival may occur, (ii) the service of the customer may be completed.

The next arrival can be determined from inter-arrival time. Thus \( E_a \) is determined at time \( t_2 \). The departure time of the customer in service is obtained from service time, and thus event
$E_d$ is generated at time $t_3$. Now, both the events $E_3$ (at $t_1$) and $E_4$ (at $t_3$) are listed in chronological order, so that the simulator may recognize that the event $E_3$ occurs before $E_4$. The next event under consideration is $E_3$ at $t_1$ and at this moment $E_3$ at $t_1$ is deleted from the stored list (because of past event). The event $E_3$ at $t_1$ now generates $E_4$ at $t_3$. Since the service facility is busy, the new arriving customer $E_4$ (at $t_3$) joins a waiting line. Now, $E_3$ at $t_1$ is deleted from the list and $E_4$ at $t_3$ is considered next. At this time a customer is taken from the waiting line and departure event $E_d$ at $t_5$ is generated. This process is repeated until the entire simulated period $T$ is covered.

9. Applications:

Simulation has a large number of applications. For example, it can be used for learning about the operating characteristics of a new airplane by simulating flight conditions in a wind tunnel, on electronic or hydraulic analog models of production processes or economic systems, or on mathematical models of such real-life systems as inventory control, production scheduling, network analysis, and so on. It can also be used for planning military strategy, traffic control, management games and role playing, medical diagnosis, hospital emergency facilities, gambling and analysis, location analysis, e.g. determining optimal location for plants and warehouses, evaluation of industrial and commercial policies.

We now discuss a few applications in detail.

10. Applications to Inventory Control:

For providing efficient service to the customers, it is necessary to choose to reorder point with proper consideration of demand during lead time. If the lead time and demand of inventory per unit time both are random variables, then the simulation technique can be applied to determine the effect of alternate inventory policies on a stochastic inventory system, e.g. different combinations of order quantity and reorder point. The basic approach would be to find the probability distribution of the input and output and output functions of the past data. Then, we run the inventory system artificially by generating the future observation on the assumptions of the same distributions. The method involves a good amount of computation. But, in simple problems it is possible to generate artificial samples for future with the help of random numbers and then the entire computations are done.
with the help of desk calculator. Of course, it becomes necessary to use electronic computers for solving more complex problems.

The technique is illustrated with the help of following simple inventory problems.

Example 7. A book store wishes to carry ‘Ramayana’ in stock. Demand is probabilistic and replenishment of stock takes 2 days (i.e., if an order is placed on March 1, it will be delivered at the end of the day on March 3). The probabilities of demand are given below:

<table>
<thead>
<tr>
<th>Demand (daily)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>0.05</td>
<td>0.10</td>
<td>0.30</td>
<td>0.45</td>
<td>0.10</td>
</tr>
</tbody>
</table>

Each time an order is placed, the store incurs an ordering cost of Rs. 10 per order. The store also incurs a carrying cost of Rs. 0.50 per book per day. The inventory carrying cost is calculated on the basis of stock at the time of each day.

The manager of the book store wishes to compare two options for his inventory decision.

A: Order 5 books when the inventory at the beginning of the day plus orders outstanding is less than 8 books.

B: Order 8 books when the inventory at the beginning of the day plus orders outstanding is less than 8.

Currently (beginning of the 1st day) the store has stock of 8 books plus 6 books ordered 2 days ago and expected to arrive next day.

Using Monte-Carlo simulation for 10 cycles, recommend which option the manager should choose.

The two digit random numbers are given below:
89, 34, 78, 63, 61, 81, 39, 16, 13, 73
Solution.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.05</td>
<td>0.05</td>
<td>00-04</td>
</tr>
<tr>
<td>1</td>
<td>0.10</td>
<td>0.15</td>
<td>05-14</td>
</tr>
<tr>
<td>2</td>
<td>0.30</td>
<td>0.45</td>
<td>15-44</td>
</tr>
<tr>
<td>3</td>
<td>0.45</td>
<td>0.90</td>
<td>45-89</td>
</tr>
<tr>
<td>4</td>
<td>0.10</td>
<td>1.00</td>
<td>90.99</td>
</tr>
</tbody>
</table>

Stock in hand = 8, and stock on order = 6 (expected next day).

Option A.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>89</td>
<td>3</td>
<td>8</td>
<td>-</td>
<td>5</td>
<td>6</td>
<td>-</td>
<td>6</td>
</tr>
<tr>
<td>34</td>
<td>2</td>
<td>5</td>
<td>6</td>
<td>9</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>78</td>
<td>3</td>
<td>9</td>
<td>6</td>
<td>-</td>
<td>5</td>
<td>-</td>
<td>5</td>
</tr>
<tr>
<td>63</td>
<td>3</td>
<td>6</td>
<td>3</td>
<td>5</td>
<td>-</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>61</td>
<td>3</td>
<td>3</td>
<td>-</td>
<td>0</td>
<td>5</td>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>81</td>
<td>3</td>
<td>0</td>
<td>5</td>
<td>2</td>
<td>5</td>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>39</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>10</td>
<td>-</td>
<td>10</td>
<td>-</td>
</tr>
<tr>
<td>16</td>
<td>2</td>
<td>0</td>
<td>3</td>
<td>5</td>
<td>-</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>13</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>0</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>73</td>
<td>3</td>
<td>7</td>
<td>-</td>
<td>4</td>
<td>5</td>
<td>-</td>
<td>5</td>
</tr>
</tbody>
</table>

No. of orders = 4 (ordering cost) = 4 \times 10 = Rs.40

Closing stock of 10 days = 39, carrying cost = 39 \times 0.50 = 19.50

Option B.

<table>
<thead>
<tr>
<th>sales</th>
<th>Opt. Stock in hand</th>
<th>Receipt</th>
<th>Cl. Stock in hand</th>
<th>Opt. stock on order</th>
<th>Order Qty.</th>
<th>Cl. Stock on order</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>-</td>
<td>5</td>
<td>-</td>
<td>6</td>
<td>-</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>85</td>
<td>6</td>
<td>9</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>6</td>
<td>-</td>
<td>8</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>-</td>
<td>3</td>
<td>8</td>
<td>-</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>-</td>
<td>0</td>
<td>8</td>
<td>-</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>8</td>
<td>5</td>
<td>-</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>-</td>
<td>3</td>
<td>8</td>
<td>-</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>-</td>
<td>1</td>
<td>8</td>
<td>-</td>
<td>8</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>8</td>
<td>-</td>
<td>8</td>
<td>-</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>-</td>
<td>5</td>
<td>-</td>
<td>8</td>
<td>8</td>
</tr>
</tbody>
</table>

No. of orders = 3, ordering cost = Rs. 30.
Closing stock of 10 days = 43, carrying cost = 45 x 0.50 = Rs. 22.50. Since option B has lower cost, manager should choose option B.

Example 8. Consider an inventory situation in a manufacturing concern. If the number of sales per day is Poisson with mean 5, then generate 30 days of sales by Monte-Carlo method.

**Solution.** Here the sales follow Poisson distribution with mean equal to 5. So we calculate the probabilities for demand from 0 to 12.

The probability for sales is given by \( P(x = s) = \frac{e^{-m}m^s}{s!} \), where \( m = 5 \) (given).

Let \( x = e^x \), then by taking logarithm, we have

\[
\log x = -5 \log e = -5 \times 0.4343 = -2.1715 = -3 + 0.8285 = 3.8285
\]

\( \therefore x = 0.006738 \) (taking anti-log)

The cumulative probabilities for \( s = 0, 1, 2, ..., 12 \) are computed below:

<table>
<thead>
<tr>
<th>Value of ( s )</th>
<th>Cumulative Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>.0067</td>
</tr>
<tr>
<td>1</td>
<td>.0404</td>
</tr>
<tr>
<td>2</td>
<td>.1247</td>
</tr>
<tr>
<td>3</td>
<td>.2650</td>
</tr>
<tr>
<td>4</td>
<td>.4405</td>
</tr>
<tr>
<td>5</td>
<td>.6160</td>
</tr>
<tr>
<td>6</td>
<td>.7622</td>
</tr>
<tr>
<td>7</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>.8666</td>
</tr>
<tr>
<td>9</td>
<td>.9319</td>
</tr>
<tr>
<td>10</td>
<td>.9682</td>
</tr>
<tr>
<td>11</td>
<td>.9763</td>
</tr>
<tr>
<td>12</td>
<td>.9845</td>
</tr>
</tbody>
</table>

We can draw a cumulative distribution graph between the sales \( s \) and the cumulative probabilities.
Now, we take 30 two-digit random numbers from the random number tables and read the corresponding values of sales from the graph. These values will give us the sales for 30 days. The values are tabulated below:

<table>
<thead>
<tr>
<th>Random Sales(s)</th>
<th>Number</th>
<th>Random Sales(s)</th>
<th>Number</th>
<th>Random Sales(s)</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>02</td>
<td>81</td>
<td>07</td>
<td>46</td>
<td>04</td>
</tr>
<tr>
<td>48</td>
<td>04</td>
<td>64</td>
<td>05</td>
<td>57</td>
<td>05</td>
</tr>
<tr>
<td>01</td>
<td>00</td>
<td>79</td>
<td>06</td>
<td>32</td>
<td>03</td>
</tr>
<tr>
<td>50</td>
<td>04</td>
<td>16</td>
<td>02</td>
<td>55</td>
<td>05</td>
</tr>
<tr>
<td>11</td>
<td>02</td>
<td>46</td>
<td>04</td>
<td>95</td>
<td>09</td>
</tr>
<tr>
<td>01</td>
<td>00</td>
<td>69</td>
<td>06</td>
<td>85</td>
<td>07</td>
</tr>
<tr>
<td>53</td>
<td>05</td>
<td>17</td>
<td>03</td>
<td>39</td>
<td>04</td>
</tr>
<tr>
<td>60</td>
<td>05</td>
<td>92</td>
<td>08</td>
<td>33</td>
<td>03</td>
</tr>
<tr>
<td>20</td>
<td>03</td>
<td>23</td>
<td>03</td>
<td>09</td>
<td>02</td>
</tr>
<tr>
<td>11</td>
<td>02</td>
<td>68</td>
<td>06</td>
<td>93</td>
<td>09</td>
</tr>
</tbody>
</table>

Application to Queueing Problems

In fact, some queueing problems cannot be solved explicitly by analytical methods. In such cases, the only possible method of solution is to simulate the experiment.

The method of simulation technique for queueing problems is illustrated below.

Example 9. Records of 100 truck loads of finished hobs arriving in a department’s check-out area show the following: Checking out takes 5 minutes and checker takes care of only one truck at a time. The data is summarized in the following table:

<table>
<thead>
<tr>
<th>Truck inter-arrival time(min.)</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>17</td>
</tr>
<tr>
<td>4</td>
<td>23</td>
</tr>
<tr>
<td>5</td>
<td>23</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>7</td>
<td>5</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
</tr>
<tr>
<td>10</td>
<td>(Total =100)</td>
</tr>
</tbody>
</table>

As soon as the trucks are checked out, the truck drivers take them to the next departments. Using Monte-Carlo simulation, determine:

(a) What is the average waiting time before service?  (b) What is likely to be the longest wait?

Solution. From the given distribution of truck arriving times, we construct a cumulative probability distribution, as shown in the following table. This table enable us to select the range of random numbers for which we shall choose a prescribed value of time.
Table 17.9

<table>
<thead>
<tr>
<th>Period (length=1 min)</th>
<th>Frequency</th>
<th>Probability</th>
<th>Cumulative Prob.</th>
<th>Random Numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.01</td>
<td>0.01</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>0.04</td>
<td>0.05</td>
<td>1-4</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>0.07</td>
<td>0.12</td>
<td>5-11</td>
</tr>
<tr>
<td>4</td>
<td>17</td>
<td>0.17</td>
<td>0.29</td>
<td>12-28</td>
</tr>
<tr>
<td>5</td>
<td>31</td>
<td>0.31</td>
<td>0.60</td>
<td>29-59</td>
</tr>
<tr>
<td>6</td>
<td>23</td>
<td>0.23</td>
<td>0.83</td>
<td>60-82</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>0.07</td>
<td>0.90</td>
<td>83-89</td>
</tr>
<tr>
<td>8</td>
<td>5</td>
<td>0.05</td>
<td>0.95</td>
<td>90-94</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
<td>0.03</td>
<td>0.98</td>
<td>95-97</td>
</tr>
<tr>
<td>10</td>
<td>2</td>
<td>0.02</td>
<td>1.00</td>
<td>98-99</td>
</tr>
</tbody>
</table>

Now, the Table 17.9 becomes the basis for generating arrival and service times. A simulation work-sheet can be prepared in the following way (see Table 17.10)

First we select 20 two digit random numbers from the random number table. The first random number for the arrival time is 12. This number lies in the range (12-28), as shown in Table 17.9. Therefore, this random number indicates the arrival time at 10.04 a.m. assuming that the checking starts at 10.00 a.m. Similarly, we can work-out all the simulated arrivals and service times. Since the first truck arrives at 10.04 a.m., the checker waits for 4 minutes. This is indicated in the last column, as checker’s waiting time in Table 17.10. The checker takes 5 minutes and thus the service for first truck will end at 10.09 a.m. The next truck will arrive at 10.10 a.m. which indicates that the checker waits for 1 minute. Whenever the truck has to wait because of the checker being busy in dealing with previous truck, the waiting time is listed in the last column of Table 17.10. The similar procedure can be adopted to prepare the entire work-sheet.
Table 17.10.

<table>
<thead>
<tr>
<th>S.No.</th>
<th>Random Number</th>
<th>Inter-arrival time(min)</th>
<th>Arrival time</th>
<th>Service begins</th>
<th>Service ends</th>
<th>Waiting time for</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12</td>
<td>4</td>
<td>10.04</td>
<td>10.04</td>
<td>10.09</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>81</td>
<td>6</td>
<td>10.10</td>
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<td>10.15</td>
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</tr>
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<td>3</td>
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<td>10.15</td>
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<td>-</td>
</tr>
<tr>
<td>4</td>
<td>82</td>
<td>6</td>
<td>10.21</td>
<td>10.21</td>
<td>10.26</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
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<td>4</td>
<td>10.25</td>
<td>10.26</td>
<td>10.31</td>
<td>-</td>
</tr>
<tr>
<td>6</td>
<td>74</td>
<td>6</td>
<td>10.31</td>
<td>10.31</td>
<td>10.36</td>
<td>-</td>
</tr>
<tr>
<td>7</td>
<td>90</td>
<td>8</td>
<td>10.39</td>
<td>10.39</td>
<td>10.44</td>
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</tr>
<tr>
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<td>5</td>
<td>10.44</td>
<td>10.44</td>
<td>10.49</td>
<td>-</td>
</tr>
<tr>
<td>9</td>
<td>79</td>
<td>6</td>
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<td>10.50</td>
<td>10.55</td>
<td>1</td>
</tr>
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<td>70</td>
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<td>10.56</td>
<td>10.56</td>
<td>11.01</td>
<td>1</td>
</tr>
<tr>
<td>11</td>
<td>14</td>
<td>4</td>
<td>11.00</td>
<td>11.01</td>
<td>11.06</td>
<td>-</td>
</tr>
<tr>
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<td>59</td>
<td>5</td>
<td>11.05</td>
<td>11.06</td>
<td>11.11</td>
<td>1</td>
</tr>
<tr>
<td>13</td>
<td>62</td>
<td>6</td>
<td>11.11</td>
<td>11.11</td>
<td>11.16</td>
<td>-</td>
</tr>
<tr>
<td>14</td>
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<td>5</td>
<td>11.16</td>
<td>11.16</td>
<td>11.21</td>
<td>-</td>
</tr>
<tr>
<td>15</td>
<td>15</td>
<td>4</td>
<td>11.20</td>
<td>11.21</td>
<td>11.26</td>
<td>1</td>
</tr>
<tr>
<td>16</td>
<td>18</td>
<td>4</td>
<td>11.24</td>
<td>11.26</td>
<td>11.31</td>
<td>2</td>
</tr>
<tr>
<td>17</td>
<td>74</td>
<td>6</td>
<td>11.30</td>
<td>11.31</td>
<td>11.36</td>
<td>1</td>
</tr>
<tr>
<td>18</td>
<td>11</td>
<td>3</td>
<td>11.33</td>
<td>11.36</td>
<td>11.41</td>
<td>3</td>
</tr>
<tr>
<td>19</td>
<td>41</td>
<td>5</td>
<td>11.38</td>
<td>11.41</td>
<td>11.46</td>
<td>3</td>
</tr>
<tr>
<td>20</td>
<td>29</td>
<td>5</td>
<td>11.43</td>
<td>11.46</td>
<td>11.51</td>
<td>3</td>
</tr>
</tbody>
</table>

After completing the simulation, following information can be obtained which describe the behavior of single server counter as given below:

(i) Average waiting time of trucks before service = \( \frac{\text{Total waiting time}}{\text{Total number of arrivals}} = \frac{16}{20} = 0.8 \) minutes.

(ii) Expected longest period of waiting = 3 minutes.

(iii) Example 10. Arrivals at a service station have been found to follow Poisson process. The mean arrival rate is \( \lambda = 6 \) units per hour. Simulate five hours of arrivals at the station.

Solution. The probability of \( k \) arrivals by Poisson distribution is given by

\[ p(x = k) = \frac{e^{-\lambda} \lambda^k}{k!} \text{ when } \lambda = 6. \]

The cumulative probabilities for \( k \) or less arrivals can be easily determined from the tables of cumulative Poisson distribution for \( \lambda = 6. \).
The values may be put in the following tabular form.

**Table 17.11**

<table>
<thead>
<tr>
<th>Number of arrivals (k):</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12 or more</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability of K or less:</td>
<td>.0025</td>
<td>.0174</td>
<td>.0620</td>
<td>.1512</td>
<td>.2851</td>
<td>.4457</td>
<td>.6063</td>
<td>.7440</td>
<td>.8472</td>
<td>.9161</td>
<td>.9574</td>
<td>.9799</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

We can draw a cumulative distribution graph (similar to Fig. 17.2) between the number of arrivals (k) and the cumulative probabilities.

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We now take 25 two-digit random numbers from the random number tables and read the corresponding values of arrivals k from the graph.

**Table 17.12**

<table>
<thead>
<tr>
<th>Arrival No.</th>
<th>Random number</th>
<th>K</th>
<th>(60 \times \frac{1}{k}) minutes</th>
<th>Arrival time (min.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>3</td>
<td>20.0</td>
<td>20.0</td>
</tr>
<tr>
<td>2</td>
<td>48</td>
<td>6</td>
<td>10.0</td>
<td>30.0</td>
</tr>
<tr>
<td>3</td>
<td>01</td>
<td>2</td>
<td>30.0</td>
<td>60.0</td>
</tr>
<tr>
<td>4</td>
<td>50</td>
<td>6</td>
<td>10.0</td>
<td>70.0</td>
</tr>
<tr>
<td>5</td>
<td>11</td>
<td>3</td>
<td>20.0</td>
<td>90.0</td>
</tr>
<tr>
<td>6</td>
<td>01</td>
<td>2</td>
<td>30.0</td>
<td>120.0</td>
</tr>
<tr>
<td>7</td>
<td>53</td>
<td>6</td>
<td>10.0</td>
<td>130.0</td>
</tr>
<tr>
<td>8</td>
<td>60</td>
<td>6</td>
<td>10.0</td>
<td>140.0</td>
</tr>
<tr>
<td>9</td>
<td>20</td>
<td>4</td>
<td>15.0</td>
<td>155.0</td>
</tr>
<tr>
<td>10</td>
<td>11</td>
<td>3</td>
<td>20.0</td>
<td>174.0</td>
</tr>
<tr>
<td>11</td>
<td>81</td>
<td>8</td>
<td>7.5</td>
<td>182.5</td>
</tr>
<tr>
<td>12</td>
<td>64</td>
<td>7</td>
<td>8.5</td>
<td>191.0</td>
</tr>
<tr>
<td>13</td>
<td>79</td>
<td>8</td>
<td>7.5</td>
<td>198.5</td>
</tr>
<tr>
<td>14</td>
<td>16</td>
<td>3</td>
<td>20.0</td>
<td>218.5</td>
</tr>
<tr>
<td>15</td>
<td>46</td>
<td>6</td>
<td>10.0</td>
<td>228.5</td>
</tr>
<tr>
<td>16</td>
<td>69</td>
<td>7</td>
<td>8.5</td>
<td>236.5</td>
</tr>
<tr>
<td>17</td>
<td>17</td>
<td>4</td>
<td>15.0</td>
<td>251.5</td>
</tr>
<tr>
<td>18</td>
<td>92</td>
<td>10</td>
<td>6.0</td>
<td>257.5</td>
</tr>
<tr>
<td>19</td>
<td>23</td>
<td>4</td>
<td>15.0</td>
<td>272.5</td>
</tr>
<tr>
<td>20</td>
<td>68</td>
<td>7</td>
<td>8.5</td>
<td>281.0</td>
</tr>
<tr>
<td>21</td>
<td>46</td>
<td>6</td>
<td>10.0</td>
<td>291.0</td>
</tr>
<tr>
<td>22</td>
<td>57</td>
<td>7</td>
<td>8.5</td>
<td>299.5*</td>
</tr>
<tr>
<td>23</td>
<td>32</td>
<td>5</td>
<td>12.0</td>
<td>-</td>
</tr>
<tr>
<td>24</td>
<td>55</td>
<td>6</td>
<td>10.0</td>
<td>-</td>
</tr>
<tr>
<td>25</td>
<td>95</td>
<td>11</td>
<td>5.5</td>
<td>Five hours completed*</td>
</tr>
</tbody>
</table>

*Five hours completed*
Example 11. Hundred unemployed people were found to arrive at a one-person state-employment office to obtain their unemployment compensation cheque according to the following frequency distribution.

<table>
<thead>
<tr>
<th>Inter-arrival time(min.)</th>
<th>Frequency</th>
<th>Service Time(min.)</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>10</td>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td>3</td>
<td>20</td>
</tr>
<tr>
<td>4</td>
<td>40</td>
<td>4</td>
<td>40</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
<td>5</td>
<td>20</td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td>6</td>
<td>10</td>
</tr>
</tbody>
</table>

The Govt. is interested in predicting the operating characteristics of this one-person state-employment office during a typical operating day from 10:00 a.m. to 11 a.m. Use simulation to determine the average waiting time and total time in the system, and the maximum queue length.

Solution. Step 1. From the prescribed distribution of arrivals and service times, the random numbers can be assigned to the arrival times as tabulated below:

Table 17.13.

<table>
<thead>
<tr>
<th>Inter-arrival time(min.)</th>
<th>Frequency</th>
<th>Probability</th>
<th>Cumulative Probability</th>
<th>Random Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>10</td>
<td>0.10</td>
<td>0.10</td>
<td>00-09</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td>0.20</td>
<td>0.30</td>
<td>10-29</td>
</tr>
<tr>
<td>4</td>
<td>40</td>
<td>0.40</td>
<td>0.70</td>
<td>30-69</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
<td>0.20</td>
<td>0.90</td>
<td>70-89</td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td>0.10</td>
<td>1.00</td>
<td>90-99</td>
</tr>
</tbody>
</table>

Step 2. Proceeding as in Step 1, the random number can be assigned to service times as shown in the following table.

<table>
<thead>
<tr>
<th>Inter-arrival time(min.)</th>
<th>Frequency</th>
<th>Probability</th>
<th>Cumulative Probability</th>
<th>Random Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>10</td>
<td>0.10</td>
<td>0.10</td>
<td>00-09</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td>0.20</td>
<td>0.30</td>
<td>10-29</td>
</tr>
<tr>
<td>4</td>
<td>40</td>
<td>0.40</td>
<td>0.70</td>
<td>30-69</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
<td>0.20</td>
<td>0.90</td>
<td>70-89</td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td>0.10</td>
<td>1.00</td>
<td>90-99</td>
</tr>
</tbody>
</table>
Step 3. To develop the simulation work-sheet as shown in Table 17.15.

We select a list of random numbers for arrival time is 17, corresponding to inter-arrival time of 3 minutes. This indicates that first person arrives 3 minutes later the service window opens. So the service can be started immediately at 10.03 a.m.

The first random number in column 6 of Table 17.15 is 90, corresponding to a service time of 6 minutes. So the first person will leave the system at 10.0 a.m. But, the first person spent 6 minutes in the system, and there was no queue at the time of arrival.

<table>
<thead>
<tr>
<th>S.No</th>
<th>Random No.</th>
<th>Inter-arrival Time(min.)</th>
<th>Arrival Time</th>
<th>Service Starts</th>
<th>Random No.</th>
<th>Service Time (min.)</th>
<th>End (a.m.)</th>
<th>Waiting Time of Server (min.)</th>
<th>People (min.)</th>
<th>Time in System</th>
<th>Queue Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>17</td>
<td>3</td>
<td>10.03</td>
<td>10.03</td>
<td>90</td>
<td>6</td>
<td>10.09</td>
<td>3</td>
<td>-</td>
<td>6</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>86</td>
<td>5</td>
<td>10.08</td>
<td>10.09</td>
<td>59</td>
<td>4</td>
<td>10.13</td>
<td>-</td>
<td>1</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>84</td>
<td>5</td>
<td>10.13</td>
<td>10.13</td>
<td>95</td>
<td>6</td>
<td>10.19</td>
<td>-</td>
<td>-</td>
<td>6</td>
<td>-</td>
</tr>
<tr>
<td>4</td>
<td>79</td>
<td>5</td>
<td>10.18</td>
<td>10.19</td>
<td>68</td>
<td>4</td>
<td>10.23</td>
<td>-</td>
<td>-</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>33</td>
<td>4</td>
<td>10.22</td>
<td>10.23</td>
<td>51</td>
<td>4</td>
<td>10.27</td>
<td>-</td>
<td>1</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>55</td>
<td>4</td>
<td>10.26</td>
<td>10.27</td>
<td>82</td>
<td>5</td>
<td>10.32</td>
<td>-</td>
<td>-</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>06</td>
<td>2</td>
<td>10.28</td>
<td>10.32</td>
<td>72</td>
<td>5</td>
<td>10.37</td>
<td>-</td>
<td>4</td>
<td>9</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>42</td>
<td>4</td>
<td>10.32</td>
<td>10.37</td>
<td>01</td>
<td>2</td>
<td>10.39</td>
<td>-</td>
<td>-</td>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>9</td>
<td>93</td>
<td>6</td>
<td>10.38</td>
<td>10.39</td>
<td>77</td>
<td>5</td>
<td>10.44</td>
<td>-</td>
<td>-</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>38</td>
<td>4</td>
<td>10.42</td>
<td>10.44</td>
<td>80</td>
<td>5</td>
<td>10.49</td>
<td>-</td>
<td>-</td>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>11</td>
<td>58</td>
<td>4</td>
<td>10.46</td>
<td>10.49</td>
<td>84</td>
<td>5</td>
<td>10.54</td>
<td>-</td>
<td>3</td>
<td>8</td>
<td>1</td>
</tr>
<tr>
<td>12</td>
<td>71</td>
<td>5</td>
<td>10.51</td>
<td>10.54</td>
<td>19</td>
<td>3</td>
<td>10.57</td>
<td>-</td>
<td>-</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>13</td>
<td>74</td>
<td>5</td>
<td>10.56</td>
<td>10.57</td>
<td>34</td>
<td>4</td>
<td>11.01</td>
<td>-</td>
<td>1</td>
<td>5</td>
<td>1</td>
</tr>
</tbody>
</table>

The second person is associated with the random number 86, which indicates that the arrival time is 5 minutes. Since the service of first person ended at 10.09 a.m., the next person has to wait for 1 minute before the starting of his service. The next service time is associated with random number 59 indicating that the service will last for 4 minutes (Table 17.14). Since the server is busy, the number of people waiting in the queue is shown in the column ‘queue-length’. The simulation continues in the like manner until the closing hours.

Step 4. To compute the required information from the simulation-sheet.

(i) Average waiting time of people before service = \[\frac{\text{Total waiting time of the people}}{\text{Total number of arrivals}}\] = \[\frac{23}{13}\] = 1.77 minutes.
(ii) Average queue length = \( \frac{\text{Total number of people in the queue}}{\text{Total number of arrivals}} = \frac{12}{13} \approx 1 \) (nearly)

(iii) Average time the person spends in the system
\[ = \text{Average service time} + \text{Average waiting time before the service} \]
\[ = 4.46 + 1.77 = 6.23 \text{ minutes}. \]